COMPLEX NUMBERS – WORKSHEET #2

COURSE/LEVEL
NSW Secondary High School Year 12 HSC Mathematics Extension 2.

TOPIC
Complex Numbers: Geometric representation of complex numbers as points and vectors. (Syllabus Ref: 2.2, 3.3)

1. If $z = 3 + i$ and $w = 2 - i$ find

   (i) $z\overline{w} + iz$ (ii) $\left|\dfrac{w}{\overline{z}}\right|$ (iii) $\arg(w - z)$ (to the nearest degree)

2. Find the modulus and argument of $\dfrac{1+i}{\sqrt{3}-i}$.

3. If $z = 2 + 2i$, write the following in modulus-argument form.

   (i) $\overline{z}$ (ii) $z\,\overline{z}$ (iii) z^2 (iv) $\dfrac{1}{z}$

4. Express each of the following in the form $r(\cos\theta + i\sin\theta)$.

   (i) $1 + i$ (ii) $1 - i$ (iii) $\sqrt{3} + i$ (iv) $\sqrt{3} - i$

   Multiply each of these numbers by i and express the resulting complex numbers in the form $r(\cos\theta + i\sin\theta)$. What relation can you observe between $\arg z$ and $\arg iz$ in the above cases?

5. Let $z = a + ib$ where $a^2 + b^2 \neq 0$.

   (i) Show that if $\operatorname{Im}(z) > 0$ then $\operatorname{Im}(\frac{1}{z}) < 0$. (ii) Prove that $\left|\dfrac{1}{z}\right| = \dfrac{1}{|z|}$.

6. If a is any complex number and z is such that $|z| = 1$ $(z \neq a)$, show that $\left|\dfrac{z-a}{\overline{a}\,z-1}\right| = 1$.

7. If $|z_1 + z_2| = |z_1| + |z_2|$, show both algebraically and geometrically, that $\arg z_1 = \arg z_2$.

8. Show that $|z_1 + z_2| \geq ||z_1| - |z_2||$.

9. Show that for any two complex numbers z_1 and z_2,

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$$

   Interpret this result geometrically. (Hint: Use $z\overline{z} = |z|^2$)

10. If z_1 and z_2 are complex numbers such that $|z_1 - z_2| \leq \frac{1}{2}|z_1|$, prove that $|z_2| \geq \frac{1}{2}|z_1|$ and $|z_1 + z_2| \geq \frac{3}{2}|z_1|$

11. If $|z - a| = r$, show that

$$z\overline{z} - a\overline{z} - \overline{a}z + a\overline{a} - r^2 = 0.$$

12. Let z_1 and z_2 be two complex numbers, where

$$z_1 = -2 + i \text{, and}$$

$$|z_2| = 3 \text{ and } \arg z_2 = \frac{\pi}{3}$$

(i) On an Argand diagram plot the points A and B to represent the complex numbers z_1 and z_2.

(ii) Plot the points C and D representing the complex numbers $z_1 - z_2$ and $i z_2$, respectively. Indicate any geometric relationships between the four points A, B, C and D.

13. In the complex plane the points P_1, P_2 and P_3 represent the complex numbers z_1, z_2 and z_3 respectively. If P_1', P_2' and P_3' represent the numbers $z_2 + z_3$, $z_3 + z_1$ and $z_1 + z_2$ respectively, show that the triangles $P_1 P_2 P_3$ and $P_1' P_2' P_3'$ are congruent.

14. The points *A*, *B*, *C* and *D* on an Argand diagram represent the complex numbers $2 + 2i$, 4, $6 + 2i$ and $4 + 4i$ respectively. Prove that $ABCD$ is a square and find the complex number represented by the intersection of the diagonals.

15. The centre of a square is at the point $z_1 = 1 + i$ and one of the vertices is at the point $z_2 = 1 - i$. Find the complex numbers which correspond to the other vertices of the square.

16. *ABCD* is a square in the Argand diagram (where the vertices are labelled anti-clockwise). $z_1 = 2 + 2i$ is represented by the vertex *A* and $z_2 = -1 + i$ is represented by the vertex *B*. Find:

(i) the complex number which represents the vertex diagonally opposite vertex *A*.

(ii) the length of the square's diagonal.

17. The complex number $z = 3 - 4i$ has two square roots z_1 and z_2. Find z_1 and z_2 in the form $a + ib$. Show that the three points representing z, z_1 and z_2 on an Argand diagram, are the vertices of a right angled triangle.

18. (a) *ABCDEF* is a regular hexagon on an Argand diagram. The centre of the hexagon is at the origin *O*. The vertex *A* represents the complex number *z*. Find the complex number represented by *B* (where *B* is the nearest vertex to *A* in the anti-clockwise direction.)

(b) If the hexagon is now rotated about *O* in an anti-clockwise direction by $45°$. Find the complex number represented by the new position of point *B*.