COMPLEX NUMBERS – WORKSHEET #3

COURSE/LEVEL

NSW Secondary High School Year 12 HSC Mathematics Extension 2.

TOPIC

Complex Numbers: Powers and roots of complex numbers. (Syllabus Ref: 2.4)

1. Prove by induction that $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ for all integers $n \geq 1$.

2. (i) Express $\sqrt{3} + i$ in modulus-argument form.

   (ii) Hence evaluate $\left(\sqrt{3} + i\right)^6$.

3. (i) Write $\omega = \dfrac{1 + i\sqrt{3}}{2}$ in polar (that is, modulus-argument) form.

   (ii) Use De Moivre's Theorem to show that $\omega^3 = -1$.

   (iii) Hence calculate ω^{10}.

4. Evaluate $\left(1 + \sqrt{3}i\right)^{10}$ in the form $x + iy$.

5. (i) If $z = 2\left(\sin\dfrac{\pi}{6} + i\cos\dfrac{\pi}{6}\right)$ evaluate z^6.

   (ii) Plot, on an Argand diagram, all complex numbers that are solutions of $z^6 = -64$.

6. Express each of the following numbers in the form $a + ib$ where a and b are real.

   (i) $\dfrac{(1+2i)^2 - (1-i)^3}{(3+2i)^3 - (2+i)^2}$ (ii) $\dfrac{(1+i)^9}{(1-i)^7}$ (iii) $\sqrt[4]{2 - i\sqrt{12}}$

   (iv) $\left(az^2 + bz\right)\left(bz^2 + az\right)$, where $z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and a and b are real.

7. Let θ be a real number and consider $(\cos\theta + i\sin\theta)^3$.

   (a) Prove that $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$.

   (b) Find a similar expression for $\sin 3\theta$.

8. Factorise $z^5 + 1$ into real linear and quadratic factors. Hence or otherwise show that

$$\cos\frac{\pi}{5}\cos\frac{2\pi}{5} = \frac{1}{4},$$

$$\sin\frac{\pi}{5}\sin\frac{2\pi}{5} = \frac{\sqrt{5}}{4}$$

9. If ω is a non-real root of the equation $x^3 - 1 = 0$ then

   (i) Show that ω^2 is also a root.

   (ii) Deduce that $1 + \omega + \omega^2 = 0$

10. (a) Solve the equation $z^6 + 1 = 0$, giving roots in the form $a + ib$. Show these roots on an Argand diagram.

(b) Factorise $z^6 + 1$ into real quadratic factors.

11. Prove that

$$\sin\theta + \sin 2\theta + \ldots + \sin n\theta = \frac{\sin\frac{n\theta}{2}\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}$$

and

$$1 + \cos\theta + \cos 2\theta + \ldots + \cos n\theta = \frac{\cos\frac{n\theta}{2}\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}$$

12. (a) Write down the modulus-argument form of $(1+i)^n$.

(b) Expand $(1+i)^n$ using the binomial theorem

(c) Using parts (a) and (b) show that

$$1 - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \cdots = 2^{\frac{n}{2}} \cos\frac{n\pi}{2},$$

$$\binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \binom{n}{7} + \cdots = 2^{\frac{n}{2}} \sin\frac{n\pi}{2}.$$

13. (a) If $z = r(\cos\theta + i\sin\theta)$ find an expression for $z^n + z^{-n}$.

(b) Expand $\left(z^1 + z^{-1}\right)^4$ and using the above result express your answer in the form $A\cos 4\theta + B\cos 2\theta + C$.

(c) Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^4\theta \, d\theta$.

14. (a) If ω is the complex root of $z^5 - 1 = 0$ with the smallest positive argument, show that ω^2, ω^3 and ω^4 are the other roots.

(b) Show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$.

(c) The quadratic polynomial $z^2 - (\alpha + \beta)z + \alpha\beta = 0$ has roots α and β. Use this fact to find the quadratic equation whose roots are $\alpha = \omega + \omega^4$ and $\beta = \omega^2 + \omega^3$.