JENSEN'S INEQUALITY IN DETAIL AND S-CONVEX FUNCTIONS

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ABSTRACT. We here study the inequality by Jensen for the case of S-convexity.

1. INTRODUCTION

2. NOTATIONS AND DEFINITIONS

2.1. **Notations.** We use the symbology defined in [PINHEIRO2006]:

- K_s^1 for the class of $S-$convex functions in the first sense, some s;
- K_s^2 for the class of $S-$convex functions in the second sense, some s;
- K_0 for the class of convex functions;
- s_1 for the variable S, $0 < s_1 \leq 1$, used for the first type of S-convexity;
- s_2 for the variable S, $0 < s_2 \leq 1$, used for the second type of s-convexity.

Remark 1. The class of 1-convex functions is simply a restriction of the class of convex functions, which is attained when $X = \Re_+$,

$$K_1^1 \equiv K_1^2 \equiv K_0.$$

2.2. **Definitions.** We use the definitions presented in [PINHEIRO2006] in what regards $S-$convexity, as well as in [PINHEIRO2008], in what regards convexity:

Definition 1. *$f : I- > \Re$ is considered convex iff*

$$f[\lambda x + (1-\lambda)y] \leq \lambda f(x) + (1-\lambda)f(y)$$

$\forall x, y \in I, \lambda \in [0,1]$.

Definition 2. *A function $f : X- > \Re$ is said to be s_1-convex if the inequality*

$$f(\lambda x + (1-\lambda^s)^{\frac{1}{s}} y) \leq \lambda^s f(x) + (1-\lambda^s)f(y)$$

holds $\forall \lambda \in [0,1]$; $\forall x, y \in X$; $X \subset \Re_+$.

Remark 2. If the complementary concept is verified, then f is said to be s_1-concave.

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Definition 3. *A function $f : X -> \Re$ is called s_2-convex, $s \neq 1$, if the graph lies below a 'bent chord' (L) between any two points, that is, for every compact interval $J \subset I$, with boundary ∂J, it is true that*

$$sup_J(L - f) \geq sup_{\partial J}(L - f).$$

Definition 4. *A function $f : X -> \Re$ is said to be s_2-convex if the inequality*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y)$$

holds $\forall \lambda \in [0, 1]$; $\forall x, y \in X$; $X \subset \Re_+$.

Remark 3. If the complementary concept is verified, then f is said to be s_2-concave.

3. Preliminaries

3.1. **Jensen's Inequality.** We find Jensen's inequality defined the following way (see [$PEARCE$2002], for instance, page 21):
For a real convex function Φ, numbers x_i in its domain, and positive weights a_i, Jensen's inequality can be stated as:

$$\Phi\left(\frac{\sum_{n=1}^{p} a_i x_i}{\sum_{n=1}^{p} a_i}\right) \leq \frac{\sum_{n=1}^{p} a_i \Phi(x_i)}{\sum_{n=1}^{p} a_i},$$

with the inequality reversed if Φ is concave. As a particular case, if the weights a_i are all equal to unity, then

$$\left(\frac{\sum_{n=1}^{p} x_i}{p}\right) \leq \frac{\Phi(x_i)}{p}.$$

In the vast majority of the literature, however (see [YEH1999], for instance), Jensen's inequality is mentioned without the denominator. As incredible as it may seem, both [$PEARCE$2002] and [YEH1999] seem to think that the statement above is the same as Jensen's inequality, provided one states that the sum from the denominator is one...
Once more, an equivocated assertion. Of course it is not the same. Basically, if we tie one coefficient to the other with the condition that the sum is one, we cannot state that this is the same as a generic situation, where the coefficients are allowed to hold any sum they like...
Of course there will be several cases which will not be included in Jensen's inequality which are included here.

Proof. Steps:

- The convexity condition would give us easily the broader inequality, once all which is made is forcing the sum of the coefficients to be one via division by the total of the coefficients accompanying the top domain members.
- Proved.

□

3.2. Extension of broader inequality to $S-$convexity.

s_1-convexity

For a real s_1-convex function Φ, numbers x_i in its domain, and positive weights a_i, we have:

$$\Phi\left(\frac{\sum_{n=1}^{p} a_i^{\frac{1}{s}} x_i}{\left(\sum_{n=1}^{p} a_i\right)^{\frac{1}{s}}}\right) \leq \frac{\sum_{n=1}^{p} a_i \Phi(x_i)}{\sum_{n=1}^{p} a_i}.$$

Proof. Steps:

- Suffices, once more, applying the definition to the forced 'sum-one' coefficients.
- Proved.

□

s_2-convexity

For a real s_2-convex function Φ, numbers x_i in its domain, and positive weights a_i, we have:

$$\Phi\left(\frac{\sum_{n=1}^{p} a_i x_i}{\sum_{n=1}^{p} a_i}\right) \leq \frac{\sum_{n=1}^{p} a_i^s \Phi(x_i)}{\left(\sum_{n=1}^{p} a_i\right)^s}.$$

Proof. Steps:

- Suffices, once more, applying the definition to the forced 'sum-one' coefficients.
- Proved.

□

Remark 4. If the reverse direction of the inequality is verified, we state that the function is $S-$concave instead.

4. References

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