A note on Mercer's results and settlement of the definition of $S-$convex sequences

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Remark 1. Short title: Mercer's inequality and S-convex sequences.

Abstract

In this one more precursor paper, we wish to settle the concept of $S-$convex sequence, as a main target. Second, we wish to improve the wording of Mercer's work, on convex sequence inequalities, as well as fix a few of his results.

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1 Introduction

From [3-MERCER1], we learn that if 'a', 'b', and n are natural numbers, zero not allowed as a value for any of the variables involved, then:

$$\frac{1}{n+1}[a^n + a^{n-1}b + ... + b^n] \geq \left(\frac{a+b}{2}\right)^n.$$

The above equates to:

$$\frac{1}{n+1}\sum_{m=0}^{n} a^{n-m}b^m \geq \left(\frac{a+b}{2}\right)^n,$$

In [1-HARBER], one finds the following proof steps:

- Assume $a \geq b$;
- Divide all by a^n.

  Here, according to our development so far, we then get:

  $$\frac{1}{n+1}\sum_{m=0}^{n} a^{-m}b^m \geq \left(\frac{a+b}{2a}\right)^n;$$

- Set $x = \frac{b}{a}$. According to our notation, we then get:

  $$\frac{1}{n+1}\sum_{m=0}^{n} x^m \geq \left(\frac{1+x}{2}\right)^n; \qquad (Inequality \;\; 1);$$

- From $[1-HABER]$, we are reminded that: $(1+x)^n = \sum_0^{[n/2]*}(x^i + x^{n-i})$ and $\sum_0^n x = \sum_0^{[n/2]*}(x^i + x^{n-1})$ and, from here, we end up at Inequality 2;
- Inequality 1 is then equivalent to Inequality 2, and working with one of them is the same as working with the other.

From [3-MERCER1], we learn the following lemma:

Lemma 2. *For a sequence $\{\beta_n\}$, which is non-increasing, and a sequence $\{\alpha_v\}$, whose sum from 0 to n is null, which is also non-increasing, and if the latter is ordered in a manner such that all positive members precede the negative ones, then, for the sequence $\{\alpha_v\beta_v\}$, the sum of all members, from 0 to n gives us a non-negative value.*

The proof of this lemma is quite easy. Once there must be same sum, in modulus, for negative and positive parts of the second sequence, if the positive ones are multiplied by the highest in value figures of the first sequence, it should be the case that the result can only be either zero or positive.
Mercer ([4-MERCER2]) also claims to have produced the following result, which will be proven to be an equivocated development from both an enthymeme contained in the main theorem involved and a mathematical impossibility:

Theorem 2.1. *Let $\{u_v\}_{v=0}^{n}$ be a convex sequence[1]. Then*

$$\frac{1}{n+1}\sum_{v=0}^{n} u_v \geq \frac{1}{2^n}\sum_{v=0}^{n}\binom{n}{v}u_v. \quad \textit{(Inequality 2)}$$

Below, we go through the process of scrutinizing Mercer's work, going step by step of it, regarding his claimed proof of the theorem just stated. These are the steps found at Mercer's paper:

[1] Notice how interesting it is the notation used by both Haber and Mercer. One wonders why complicating. The natural thing to do with a sequence would be starting from $v = 1$ instead...

- Put $Q = [\frac{n}{2}]$;

  Problem O: we do not really know what symbol is this one, used by Mercer. Most obvious inference is that there was a typo here and it should actually read $Q = \lfloor \frac{n}{2} \rfloor$, once this is compatible with the equality mentioned by him, connected, according to himself, to the inequality he wants to prove, as well as that to which Lemma 2 would apply.

- Write

$$\sum_{v=0}^{Q_*} \gamma_v = \begin{cases} \gamma_0 + \gamma_1 + ... + \gamma_Q, \text{ case n is odd} \\ \gamma_0 + \gamma_1 + ... + \gamma_{Q-1} + \frac{1}{2}\gamma_Q, \text{ case n is even} \end{cases} ;$$

- The previous step allow us to produce the following equality:

$$\frac{1}{n+1} \sum_{v=0}^{n} u_v - \frac{1}{2^n} \sum_{v=0}^{n} \begin{pmatrix} n \\ v \end{pmatrix} u_v = \sum_{v=0}^{Q_*} c_v [u_v + u_{n-v}],$$

  where $c_v = \frac{1}{n+1} - \frac{1}{2^n} \begin{pmatrix} n \\ v \end{pmatrix}$;

  (**Problem P: Notice that, contrary to p.** 611, **from [MERCER2],** c_v **does not sum zero in** $\sum_{v=0}^{Q_*} c_v$. **Easy counter-examples are found** ($n = 3$, **for instance)).**

- Notice that $\{c_v\}$ is non-decreasing, rather than non-increasing, once supposing this is the case, that is, the member number n is less than, or equal to, the member number $n+1$, leads to the following development:

$$\frac{1}{n+2} - \frac{1}{2^{n+1}} \begin{pmatrix} n+1 \\ v \end{pmatrix} \geq \frac{1}{n+1} - \frac{1}{2^n} \begin{pmatrix} n \\ v \end{pmatrix}$$

$$\iff \frac{-1}{(n+1)(n+2)} - \frac{1}{2^{n+1}}\begin{pmatrix} n+1 \\ v \end{pmatrix} + \frac{1}{2^n}\begin{pmatrix} n \\ v \end{pmatrix} \geq 0.$$

Remember that $\begin{pmatrix} n \\ v-1 \end{pmatrix} + \begin{pmatrix} n \\ v \end{pmatrix} = \begin{pmatrix} n+1 \\ v \end{pmatrix}$. With this, we have:

$$\tfrac{-1}{(n+1)(n+2)} - \tfrac{1}{2^{n+1}}\left[\begin{pmatrix} n+1 \\ v \end{pmatrix} - 2\left[\begin{pmatrix} n+1 \\ v \end{pmatrix} - \begin{pmatrix} n \\ v-1 \end{pmatrix}\right]\right] \geq 0$$

$$\iff \frac{-1}{(n+1)(n+2)} + \frac{n!}{2^{n+1}}\left[\frac{n-2v+1}{(n-v+1)!v!}\right] \geq 0,$$

which is negative, trivially, if $(n-2v+1)$ is, as well, that is, only if $v \geq \frac{n+1}{2} \geq \frac{n}{2}$. Because there is a restriction imposed to all, that v gets split into sets going to $\frac{n}{2}$, then it is always true;

- The sequence $\{u_v\}$ being convex, with sum zero, and non-increasing, one could apply the previously mentioned Lemma to it, possibly. However, suppose a sequence is both convex and non-increasing. Let's take the definition of convex sequence into consideration:

$$2a_{n+1} \leq a_n + a_{n+2}, \forall n \in N.$$

If the sequence is also non-increasing, it is true that $a_n \geq a_{n+1} \geq a_{n+2}$. Suppose, then, that $a_n = a_{n+2} + \delta$, and that $a_{n+1} = a_{n+2} + \delta_1$. This way, all the previously stated would imply

$$2\delta_1 \leq \delta, \forall n \in N,$$

what is possible.

Now we also need to prove that such a sequence may have sum zero. To hold sum zero, some of its members will be positive, whilst others will be negative, trivially. However, assuming that $a_k < 0$, $a_{k+1} > 0$, and willing to find the same situation as in the previous statement, that is: $a_n = a_{n+2} + \delta$, and $a_{n+1} = a_{n+2} + \delta_1$, plus $2\delta_1 \leq \delta$, we would have $a_{n+2} + \delta < 0$ and $a_{n+2} + \delta_1 > 0$, then $-\delta_1 < a_{n+2} < -\delta$, what makes $-\delta_1 + \delta < a_n < 0$ and $0 < a_{n+1} < -\delta + \delta_1$. With this: $\delta_1 < a_n < 0$, what is inconsistent, once δ_1 is either zero or positive. It would also be true that $0 < a_{n+1} < -\frac{\delta}{2}$, what is also inconsistent;

We have then just proved that a sequence cannot be both non-increasing and convex, at least if its members are different from each other.

- On the other hand, c_v does not have sum zero either. We need at least one sequence with sum zero for Mercer's result to work. Therefore, it would have to then be the convex sequence attached. However, it also does not have sum zero, so that the theorem he claims to make use of does not fit there at all, and he does not prove what he would like to have proved with what he thinks to be an extension, but is not.

- As for the second theorem found in Mercer's paper, its proof gets finalized by Mercer's written statement (with minor editing), on p.3 of $[3 - MERCER1]$: 'once (u_k) is convex, and $\{c_k\} \subset \Re_+$, we arrive at the result $\sum_0^n a_k u_k \geq 0$'. Comparing the definition of convex sequence with $X = u_{k+2} - 2u_{k+1} + u_k$, one reaches the conclusion that X can only be non-negative. Because he claims that c_k would also be non-negative, then $\sum_0^n a_k u_k$ will also be and, therefore, according to the reasoning exposed in $p.3$ of $[3 - MERCER1]$, so is $\sum_0^n a_k x^k$, as intended.

- The conclusion is then that the result, which gets to being proven sound, by Mercer, is 'for any convex sequence $\{u_k\}$, and any set of non-negative coefficients c_k, attained after division of main polynomial by its two prime factors, containing root one, is made, these being only coefficients found left in the resulting polynomial, then, if we call that $\sum_o^n a_k x^k$, we hold this result as being non-negative', as proved recently in this paper. There is no mention to the possibly negative part of the coefficients after division by prime factors is made, so that the main theorem intended is a fake. In reality, Mercer worked with the premise 'b_k is zero'. However, it is still true, as it reads there, that: if (u_k) is convex and $\{c_k\} \subset \Re_+$, then $\sum_0^n a_k u_k = \sum_0^{n-2} c_k(u_{k+2} - 2u_{k+1} + u_k)$ is verified if and only if a_k is non-negative, all the way through. Unfortunately, this is not the same as stating that the product formed by elements of a convex sequence, and positive constants, is non-negative. In fact, counter-examples to this statement are easily found (for instance, $(1, 2, 3)$ is a convex sequence, once $4 \leq 1 + 3$. And so is $(1, -2, 3)$, once $-4 \leq 4$. However, choose your $c_k s$ to be $\{0, 2, 1\}$, and in multiplying them we get $(0, -4, 3)$, and the sum of these elements is, unfortunately, clearly negative, contradicting what could have been a theorem by Mercer, but there is no mercy there: it cannot be such). The small, perhaps irrelevant, result, regarding convex sequences, cannot be extended to $S-$convex sequences because the direction of the inequality will change for those with $S \neq 1$.

- The observation regarding the values, used as coefficients, being positive, for the theorem mentioned by Mercer to apply, is actually made in a paper from 2005 from the same Jipam.

Mercer's work is motivation for the introduction of the definition of $S-$convex sequences, the main objective of this paper.
We here follow this order of presentation:

1. Definitions and Notations used by us to deal with $S-$convexity in general (coherence test for results for sequences);

2. Definition of both convex and $S-$convex sequences, the second result being our novelty;

3. Conclusion.

3 Notation and Definitions

We use the symbology defined in [5-PINHEIRO]:

- K_s^1 for the class of $S-$convex functions in the first sense, some s;
- K_s^2 for the class of $S-$convex functions in the second sense, some s;
- K_0 for the class of convex functions;
- s_1 for the variable S, $0 < s_1 \leq 1$, used for the first type of S-convexity;
- s_2 for the variable S, $0 < s_2 \leq 1$, used for the second type of s-convexity.

Remark 2. The class of 1-convex functions is simply a restriction of the class of convex functions, which is attained when $X = \Re_+$,

$$K_1^1 \equiv K_1^2 \equiv K_0.$$

We use the definitions presented in [5-PINHEIRO]:

Definition 4. *A function $f : X- > \Re$ is said to be s_1-convex if the inequality*

$$f(\lambda x + (1-\lambda^s)^{\frac{1}{s}} y) \leq \lambda^s f(x) + (1-\lambda^s) f(y)$$

holds $\forall \lambda \in [0,1]$; $\forall x, y \in X$; such that $X \subset \Re_+$.

Remark 3. If the complementary concept is verified, then f is said to be s_1-concave.

Definition 5. *A function $f : X- > \Re$ is called s_2-convex, $s \neq 1$, if the graph lies below a 'bent chord' (L) between any two points, that is, for every compact interval $J \subset I$, with boundary ∂J, it is true that*

$$sup_J(L-f) \geq sup_{\partial J}(L-f).$$

Definition 6. *A function $f : X- > \Re$ is said to be s_2-convex if the inequality*

$$f(\lambda x + (1-\lambda) y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds $\forall \lambda \in [0,1]$; $\forall x, y \in X$; such that $X \subset \Re_+$.

Remark 4. If the complementary concept is verified, then f is said to be s_2-concave.

7 $S-$convex sequences and Convex sequences

7.1 Convex sequences definition

A convex sequence is defined as a sequence where

$$2a_{n+1} \leq a_n + a_{n+2}, \forall n \in N, \quad (1)$$

or, according to the source, for better reading, putting $\delta a_n = a_n - a_{n+1}$ and $\delta^2 a_n = \delta a_n - \delta a_{n+1}$, we then may reduce all to

$$\delta^2 a_n \geq 0, n \in N.$$

To be reassured of this definition, please see [2-KUDRYAVTSEV].

7.2 Main result: defining $S-$convex sequences

Theorem 7.1. *An S-convex sequence is defined as a sequence where*

$$2^s a_{n+1} \leq a_n + a_{n+2}, \forall n \in N, \quad (2)$$

if dealing with K_s^2, <u>or</u>

$$2a_{n+1} \leq a_n + a_{n+2}, \forall n \in N, \quad (3)$$

if dealing with K_s^1.

Proof. We take the definitions in consideration:

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y).$$

We now make the image to the left correspond to the midpoint image, and call it a_{n+1}. To the midpoint image, a value of $\lambda = 0.5$ corresponds. With it, we get 0.5^s to the right side, whose members we call a_n and a_{n+2}, with all coherence that is possible to have. Therefore: $2^s a_{n+1} \leq a_n + a_{n+2}$, a. w. s.

$$f(\lambda x + (1-\lambda^s)^{\frac{1}{s}} y) \leq \lambda^s f(x) + (1-\lambda^s) f(y).$$

We now look for the midpoint of the sum of the images, because there is where the balance is found. This point is attained when $\lambda = 0.5^{\frac{1}{s}}$. We then call $f(x)$, a_n, and $f(y)$, a_{n+2}. We now must reflect this in the left side of the inequality as well. Replacing λ with its value, we get $0.5^{\frac{1}{s}}(x+y)$. As s varies from open zero to closed one, we get 0 and 0.5 as boundaries for the position of the element to the left side in the domain. Half is ideal, for it is superior quote. Therefore: $2a_{n+1} \leq a_n + a_{n+2}$, a. w. s. $\square$

8 Conclusion

In this work, we think we have nullified the claimed extension of results made by Mercer in what regards the works of Haber. We also think we have presented the best definition for $S-$convex sequences as possible.

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