Hudzik and Maligranda's S-convexity as a local approximation to convex functions

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Abstract

In this work, we establish, for the first time in the mathematical history, a detailed analysis of the geometric features of $S-$convexity in the second sense as first seen, in the broadest scientific media, in the hands of Hudzik and Maligranda, in 1994, which had definitions re-written by us, with a more convex-like looks, since 2001. We truly think $S-$convexity might become a very useful tool for any field involving Real Analysis. That is because it allows us to deal with different shapes of curves, which approach convex functions, locally, and from above. In this sense, a huge family of curves is found 'orphan' of good mathematical care. With our work, this family gets to be included in the paradise of 'nice' curves to work with, where convex ones are found.

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1 Introduction

It is widely known that convexity has got many applications, in many different areas, including optimization. It also seems that optimization is interesting for many different human fields. With this paper, we try to make $S-$convexity 'palatable' to uses in optimization, just like convexity is.

In this precursor paper, which opens way for optimization to make use of 'approximation' functions to convex functions, we provide the optimization worker with geometric tools to understand the concept of $S-$convexity as well as to use the 'limiting curve' measurement to both classify a curve as $S-$convex, determining the value of S, which would lie between 0 and 1, and start reasoning of theorems on 'approximating' convex functions from above (K_s^2).

We have already obtained a few pages of good analytical results, some of them still formally unpublished, on $S-$convexity and, although we do not intend to work on optimization theories, or computational theories, just opening way for researchers from these fields to have their additional findings (we do intend to progress with our analytical findings and we might come up with a few theorems for local approximations to convexity, studying what goes on in the neighborhood of $S-$convexity).

This paper is focused only in geometric results, due to their relevance and importance per se.

In the second section, we present our last and, hopefully, final, form of defining $S-$convexity[1], with all proposed symbology.

In the third section, we introduce the concept of 'limiting curve', which is going to distinguish curves that are $S-$convex from those that are not.

In the fourth section, we write about how the choice of S affects the limiting curve.

We finish our work with a short summary with the most meaningful com-

[1]One must pay attention here that Hudzik and Maligranda, in 1999, mentioned two kinds of convexity. S_1, as we decided to name it, is usually disregarded by literature for the not-so-easy similarity with any sort of known, or relevant, functions properties (we will, however, try to change this as well.

munications.

2 Definitions and symbology

Definition of $S-$convex functions [2]

Definition 1. *A function $f: X- > \Re \in C^1$ is said to be S_2-convex if the inequality*

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds $\forall \lambda \in [0,1]$, $\forall x, y \in X$ such that $X \subset \Re_+$.

Remark 1. f is called s_2-concave if $-f$ is s_2-convex.

Definition 2. *The function $(f : X- > \Re_f)$ [3], f continuous, is called convex if the inequality*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

holds $\forall \lambda \in [0,1]$, $\forall x, y \in X$.

Refinement on the alternative geometrical definition of convexity

Definition 3. *f is called convex if the graph lies below or on the chord between any two points, that is, for every compact interval $J \subset I$, with boundary ∂J, and every linear function L intersecting the curve representing f graphically in two points at ∂J, we have*

$$sup_J(f-L) \leq sup_{\partial J}(f-L)$$

One calls f concave if $-f$ is convex [2].

Proposed Geometrical definition for $S-$convexity [5]

Theorem 1. *f is called s_2-convex, $s \neq 1$, $f \geq 0$, if the graph lies below a 'bent chord' (L) between any two points, that is, for every compact interval $J \subset I$, with boundary ∂J, it is true that*

$$sup_J(L-f) \geq sup_{\partial J}(L-f)$$

[2] see [5]

[3] here, f means closure of $\Re$

Proof. It is easy to see that with the assumption that f belongs to C^1 there would be no doubts as to $a^s f(x)+b^s f(y)$ being a continuous smooth function. It is also easy to see that $a \leq a^s$ and $b \leq b^s$. With that, it is only possible that we get a 'bent' curve or a chord (case in which $s = 1$) as a limiting defining curve for S_2-convexity. □

3 Limiting curve

The same way that the chord joining $f(x)$ to $f(y)$-corresponding to the verification of the convexity property of the function f in the interval $[x, y]$-forms the limiting height for the curve representing f to be at, limit included, in case f is convex ([2]), and this chord is represented by $\lambda f(x)+(1-\lambda)f(y)$, all analytically expressed by the most usual statement of the property of convexity, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$, there is a 'bent' curve representing the limiting height for the curve f to be at, limit included, in case f is S-convex, $S \neq 1$ ([5]). This curve is represented by $\lambda^s f(x)+(1-\lambda)^s f(y)$, for each chosen S between 0 and 1. $S = 1$, however, is taken away from our writings obviously because if $S = 1$ one gets the property of convexity and the chord again.
It is not difficult to intuitively see that all other values of S, once $0 < S \leq 1$, are going to define limiting curves which approach the chord $\lambda f(x) + (1-\lambda)f(y)$ from above: the greater the value of S, the closer the limiting curve will be to become the mentioned chord.
The relevance of this limiting curve is extremely obvious: If we cannot precisely define it, there is no way we can classify a curve as $S-$convex by means of our naked eyes (nor even if they are not naked!). On the other hand, in getting to know the details about such a limiting curve, such as maximum height, one finds ways of excluding curves that are not of that sort immediately. In understanding the exact shape of the limiting curve, one learns how to build Geometry tools to work out $S-$convexity by hand. Easy to see how such study makes computational work easy as well.
K_s^2

From the definition, we know that this limiting curve is smooth (f is a C^1), grows from $f(x)$, up to the middle, and then decreases to the side of $f(y)$, all evenly, once $a^s f(x) = b^s f(x)$, if one considers the set of numbers fulfilling the relationship $a + b = 1$. With all this, we can be sure there is a maximum point in this curve and this maximum point should be in the middle of it.

4 Effects of S on the limiting curve

Our limiting curve has a few aspects to it: maximum height, length, and local inclination. We try to deal with each one of them in a sequence:

- **Height:** Consider a bijection between the interval $[0, 1]$ and the convex combination $\lambda f(x)+(1-\lambda)f(y)$, which is obvious if we consider that λ is between 0 and 1 (In one extreme, with $\lambda = 1$, one gets $f(x)$. In the other extreme, with $\lambda = 0$, one gets $f(y)$).

  For each λ and, therefore, each result $\lambda f(x) + (1 - \lambda)f(y)$, it corresponds a member of the domain, $(\lambda x + (1 - \lambda)y)$, whose image should be compared to that height and should prove itself to be less than, or equal to, it to allow the function to be classified as convex. One knows that $a^s f(x)+b^s f(y)$ is a continuous curve if f is continuous (f is always continuous according to our definition).

  When $a+b = 1$ is under consideration in convex functions, $af(x)+bf(y)$ naturally spans all the interval between x and y (as well as the interval between $f(x)$ and $f(y)$). On the other hand, a, as well as b, will definitely span $[0, 1]$. And a^s should be a continuous function, as well as b^s. A continuous function should take compact intervals into compact intervals. If both a^s and $f(x)$ are continuous, so is their multiplication and the sum with $b^s f(y)$.If both of them have images in compact intervals, their sum also does. With all that, we can safely state that $a^s f(x) + b^s f(y)$ will span the interval $[c, d]$, which contains $f(x)$ and $f(y)$. Because both a and b span $[0, 1]$ and a^s, b^s are 'twin' functions, the images should be exactly the same. It is then not hard to see that a^s+b^s should give us a compact $[0, 2]$ as its range of action. In the convex case, $a+b$, one can see that the image gets reduced to one, the result of that sum, via definition, being one. That results in a chord between $f(x)$ and $f(y)$, also via definition. In the exponential case, with a ratio as exponent, also easy to see that $a^s + b^s \geq 1$ and, therefore, the counter-domain is going to be a bit beyond $[0, 1]$: Say $[0, 1 + \delta]$. Trivial, that $\delta = f(s)$. One can also safely utter that the smaller the value of 's' is, the greater the counter-domain size obtained, that is, the value of δ itself. We can actually state that the greater the value of 's' is, the closer we are to $a+b$, the chord and, therefore, the smaller δ will be, approaching zero rapidly. Once the peak of $a^s + b^s$ will occur when $a = b$, it is easy to see that $2a^s$ gives us the value of δ, that is, we know 'how much' our curve can vary, at most, based on reasoning over $2a^s$.

With the limitation $a + b = 1$, we can choose $a = \frac{1}{2}$ as the peak and 2^{1-s} becomes the maximum of the interval. If $s = 1$, 1 is the answer. If $s = \frac{1}{2}$, ≈ 1.41, and so on so forth, the most interesting result, illustrating our analytical findings, being when $s = 0.1$ (we get ≈ 1.87 as an answer). It is then easy to see how as s approaches 0 we get rapidly close to 2, what clearly illustrates the fact that the limiting value for δ is 1.

We can also write things in another way to justify that fact: the maximum of the limiting s_2-curve is 2^{1-s}.

The maximum of our curve can only be found easily if $f(x)$ and $f(y)$ are at the same height, once we are only interested in the limiting curve and not in any interferences caused by the values at the extremes of the interval.

With all this in mind, it is still easy to see that a simple analysis of $a^s + b^s = \lambda^s + (1 - \lambda)^s = F(\lambda)$ will give us the value of λ for this maximum (K_s^2).

Indeed, the first derivatives of $F(\lambda)$ will lead us to $\lambda = \frac{1}{2}$, as an analytical confirmation of our earlier assertions.

One must bear in mind that our limiting curve is symmetric and, if drawn to the middle, suffices that it is then mirrored to the other side. Therefore, by drawing the curve from $\lambda = 0$ to $\lambda = \frac{1}{2}$, when the curve should reach its maximum height, we have a model for the rest of it. And, not to be forgotten, 2^{1-s} is always the maximum height for our curve.

- **Length** (How bent is the limiting curve?)**:** It is not hard to notice that an analysis of

  $f(\lambda) = \lambda^s X + (1-\lambda)^s Y = (\lambda^s X, (1-\lambda)^s Y)$, making $f(x) = X$ and $f(y) = Y$, will give us precisely the idea of how our limiting curve gets designed.

  Simple plotters, such as Maple, will let the reader know how the limiting curve looks like. It is also worth mentioning that, in the case of convex functions, the result is simply 1, a straight line, 1 meaning size of the straight line joining $f(x)$ to $f(y)$.

  The simplest way of measuring the size of the limiting curve from $f(x)$ to $f(y)$ is via arc-length:

  $$size(\lambda) = \int_0^1 \sqrt{1 + \left[\frac{df}{d\lambda}\right]^2}$$

  $$\therefore size(\lambda) = \int_0^1 \sqrt{1 + (s\lambda^{s-1} - s(1-\lambda)^{s-1})^2}$$

  $$\therefore size(\lambda) = \int_0^1 \sqrt{1 + s^2\lambda^{2s-2} + s^2(1-\lambda)^{2s-2} - 2s^2\lambda^{s-1}(1-\lambda)^{s-1}} d\lambda.$$

  Once we are interested on the effects of λ, we choose X almost equal to Y, and both almost one, once they cannot be equal by definition of $S-$convexity, but they might be almost the same, to the imperceptible difference for the naked eye...

  Notice that when $s = 0$, we get $size(\lambda) = 1$.

  When $s = 1$, we also get $size(\lambda) = 1$.

  Notice, as well, that when $\lambda = 0$ and $s = 0.5$, for instance, then $size(0) = \sqrt{\frac{5}{4}}$, and the same happens to $\lambda = 1$.

  On top, if we make use of $s = 0.5$, and experiment two symmetrical values, say 0.25 and 0.75 (in relation to the middle of the unitarian interval), we will find out that the result of the function $size(\lambda)$ will not disappoint us, confirming, once more, what we stated regarding the shape of K_s^2 members.

Of course it is also possible to prove such in a generalized manner, simply departing from an equality between what really counts in the function formula. Suffices considering any value whose distance was counted departing from zero, say own λ, such that $0 \leq \lambda \leq 0.5$, and its counter-part departing from one, $1 - \lambda$:

$$s^2\lambda^{2s-2} + s^2(1-\lambda)^{2s-2} - 2s^2\lambda^{s-1}(1-\lambda)^{s-1}$$

$$\approx$$

$$s^2(1-\lambda)^{2s-2} + s^2(\lambda)^{2s-2} - 2s^2(1-\lambda)^{s-1}(\lambda)^{s-1},$$

and there we go: we have now provided an alternative analytical deduction for our previous assertion on the shape of S-convexity!

- **Local inclination:** The local inclination of our limiting curve is easily found by means of the first derivative of $f(\lambda) = \lambda^s f(x) + (1-\lambda)^s f(y)$. Therefore, the inclination is $f_s'(\lambda) = s\lambda^{s-1}f(x) + s(1-\lambda)^{s-1}f(y)$ and varies accordingly to the value of λ. Corresponding λ, either in $f_s'(\lambda)$, or in $f(\lambda)$, with members of the interval $[x, y]$ is extremely easy, once, for each $x + \delta$, there is a $\lambda x + (1-\lambda)y$ equivalent expression (recall that x and y are fixed at the beginning of the process).

5 Conclusions

In this paper, we have provided the reader with as many geometric details as possible on the limiting curve for $S-$convexity, now equating its use to that of convexity, which is just a particular case of $S-$convexity, occurring when $S = 1$. Our work provides any reader with sufficient tools to geometrically classify a curve as $S-$convex, or not, and even determine the value of S by hand.

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