Hudzik and Maligranda's S-convexity as a local approximation to convex functions II

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Abstract

In this work, we establish, for the first time in the mathematical history, a detailed analysis of the geometric features of $S-$Convexity in the first sense as first seen, in the broadest scientific media, in the hands of Hudzik and Maligranda, in 1994, which had definitions re-written by us, with a more convex-like looks, since 2001. We truly think $S-$Convexity might become a very useful tool for any field involving Real Analysis. That is because it allows us to deal with different shapes of curves, which approach convex functions locally, and from above. In this sense, a huge family of curves is found 'orphan' of good mathematical care. With our work, this family gets to be included in the paradise of 'nice' curves to work with, where convex ones are found. One annoying 'progress' of this paper is that we actually found out why the reasoning applied in Pearce and Dragomir's work, regarding S-Convexity in first sense being formed of non-decreasing functions, is equivocated, therefore nullifying our previous weak referral of that result, now with solid argumentation.

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1 Introduction

It is widely known that Convexity has got many applications, in many different areas, including optimization. It also seems that optimization is interesting for many different human fields. With this paper, we try to make $S-$Convexity 'palatable' to uses in optimization, just like Convexity is.

In this precursor paper, which opens way for optimization to make use of 'approximation' functions to convex functions, we provide the optimization worker with Geometry tools to understand the concept of $S-$Convexity as well as to use the 'limiting curve' measurement to both classify a curve as $S-$convex, determining the value of S, which would lie between 0 and 1, and start reasoning of theorems on 'approximating' convex functions from above (K_s^1).

We have already obtained a few pages of good analytical results, some of them still formally unpublished and, even though we do not intend to work on optimization theories, or computational theories, just opening way for researchers from those fields to have their additional findings, we do intend to progress with our analysis work and we might come up with a few theorems for local approximations to Convexity, studying what goes on in the neighborhood of $S-$Convexity.

This paper is focused only in geometric results, due to their relevance and importance per se.

In the second section, we present our last and, hopefully, final, form of defining $S-$Convexity[1], with all proposed symbology.

In the third section, we introduce the concept of 'limiting curve', which is going to allow us to tell which curves are $S-$convex and which are not.

[1]One must pay attention here that Hudzik and Maligranda, in 1999, mentioned two kinds of Convexity. S_1, as we decided to name it, is usually disregarded by literature for the not-so-easy similarity with any sort of known, or relevant, functions properties (we will, however, try to change this as well).

In the fourth section, we write about how the choice of S affects the limiting curve.
We finish our work with a short summary with the most meaningful communications.

Remark 1. From here onwards, each time we come up with a new result, we place a ★ close to it to indicate it is such, and each time we prove old statements, published in the literature, so far, are equivocated, we place a ⊚ close to them to indicate it is such.

2 Definitions and symbology

Definition of $S-$convex functions[2]

Definition 1. *A function $f : X- > \Re$ is said to be s_1-convex if the inequality*

$$f(\lambda x + (1-\lambda^s)^{\frac{1}{s}} y) \leq \lambda^s f(x) + (1-\lambda^s) f(y)$$

holds $\forall \lambda \in [0,1]$*;* $\forall x, y \in X$*;* $X \subset \Re_+$.

Remark 2. If the complementary concept is verified, then f is said to be s_1-concave.

Definition 2. *A function $f : X- > \Re \in C^1$ is said to be S_2-convex if the inequality*

$$f(\lambda x + (1-\lambda) y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds $\forall \lambda \in [0,1]$, $\forall x, y \in X$ *such that* $X \subset \Re_+$.

Remark 3. f is called s_2-concave if $-f$ is s_2-convex.

Definition 3. *The function $(f : X- > \Re_f)$* [3]*, f continuous, is called convex if the inequality*

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$$

holds $\forall \lambda \in [0,1]$, $\forall x, y \in X$.

[2] see [4]
[3] here, f means closure of $\Re$

Refinement on the alternative geometrical definition of Convexity

Definition 4. *f is called convex if the graph lies below or on the chord between any two points, that is, for every compact interval $J \subset I$, with boundary ∂J, and every linear function L intersecting the curve representing f graphically in two points at ∂J, we have*

$$sup_J(f - L) \leq sup_{\partial J}(f - L)$$

One calls f concave if $-f$ is convex [2].

Proposed Geometrical definition for $S-$Convexity [4]

Theorem 1. *f is called s_1-convex, $s \neq 1$, if the graph lies below a 'bent line'(L), between any two points, that is, for every compact interval $J \subset I$, with boundary ∂J, it is true that*

$$sup_J(L - f) \geq sup_{\partial J}(L - f)$$

Proof. It is easy to see that, with the assumption that f belongs to C^1, there would be no doubts as to $a^s f(x) + b^s f(y)$ being a continuous smooth function. It is also easy to see that $a \leq a^s$ and $b \leq b^s$. However, we have previously seen, with K_s^2 (refer to [6]), that when the sum of the 'weights' on the domain points chosen is one, the image expands to almost 2. Therefore, the 'walk' covered by K_s^1 functions would have to be, at least for a specific s, close to zero, half, only, of the walk covered by K_s^2 ones. □

3 Small fixing, or refinement, in favor of accuracy, or well-posedness, on definition of S-Convexity

When working on this paper, a major issue came into play: the expressions, for both types of S-Convexity, are actually the same, and may even be written the same way, if the current wording of each definition is kept, at the exception of the constraints imposed upon a and b. However, all which happens, from one type of constraints to the other, is that one stops getting the sum half the way for s_1, has got half of the options than the other, as couples of possible points for (a, b). In $a+b = 1$, or $(a, 1-a)$, one gets $[0, 1]$ covered by only present variable. In $a^s+b^s = 1$, or $(a, (1-a^s)^{\frac{1}{s}})$, one also gets $[0, 1]$ covered. Where they differ, more remarkably, is at the curve defined between the images of the points of the domain considered. In $(a, 1-a)$, one will get, as image boundary, $(a^s, (1-a)^s)$, but in the first type, one gets $(a^s, (1-a^s))$. Notice that the first term in the couple is the same. It will vary from 0 to 1 as well, independently of the value of s. However, $(1-a^s) \leq (1-a)^s$. **(Step 1)** See[4]:

- It is true for $s = 0.5$, 2 being lowest integer which may be used as denominator.
- Suppose it is true for $s = 1/n$.
- Let's prove $s = 1/(n+1)$: $(1-a^{1/n}) \leq (1-a)^{1/n}$. Because the basis of both powers, present in the calculation, is either a ratio (non-passive of simplification), or one in two integer values (0 or 1), and assuming any of the integers will lead to immediate validation, we know that, in between integers, it is true that: $(1-a^{1/(n+1)}) \leq (1-a^{1/n})$ as well as $(1-a)^{1/n} \leq (1-a)^{1/n+1}$.
- Proved!

With this, not mattering the 's' involved, it will always be the case that s_1-Convexity will go lower than s_2-Convexity, or, at most, equal. Accompany now the following steps of proof:

- Step 1;
- Now, let's assume that Step 1 inequality is actually an equality. In this case: $(1-a^s) = (1-a)^s$. Notice that our equality could be written the following way: $\left(1-\frac{1}{m}\right)^q = 1-\left(\frac{1}{m}\right)^q$. After simplification in writing, we get: $(m-1)^q = m^q - 1$. Notice now that

[4]We prove all for $\frac{1}{n}$-sort-of-ratio. Of course real numbers will fill in the gap, so that this is not a complete proof...However, the rationals of this sort act as 'attractors', and we may extend reasoning easily using continuity and sequences, so that this is enough proof.

the only way both sides will equate is when the difference of the fractionary powers of a natural figure and the natural figure after it equals to one. For this to happen, we would better consider the ratio which allows larger expansion of basis as possible. If it is not valid in the largest, it cannot be valid, as assertion, for any other expansion...The largest expansion is achieved when $q = 0.5$. In this case, the last equality implies $m - 1 = m + 1 - 2\sqrt{m} \iff -2 = -2\sqrt{m} \iff 1 = \sqrt{m} \iff m = 1$;

- Therefore, 1 disregarded as possibility, $(1 - a^s) < (1 - a)^s$. This implies $K_s^2 >> K_s^1$.

Another interesting observation is that $f(x) = \sqrt{x}$, for instance, is not a convex function if the domain considered is $[0, 1]$ or $[0, 1] \bigcup I$, where I is any real interval of interest which allows continuity (due to the geometric definition).

However, it will be $S-$convex in both senses. See:

- K_s^2: $\sqrt{ax + (1 - a)y} \leq \sqrt{a}\sqrt{x} + \sqrt{(1 - a)}\sqrt{y} \iff ax + (1 - a)y \leq ax + (1 - a)y + 2(\sqrt{ax(1 - a)y})$, what is always true, given that $2(\sqrt{ax(1 - a)y}) \geq 0$ always for the domain of definition of the function. Therefore, $f(x) \in K_{0.5}^2$.
- K_s^1: $\sqrt{ax + (1 - \sqrt{a})^2 y} \leq \sqrt{a}\sqrt{x} + (1 - \sqrt{a})\sqrt{y} \iff ax + (1 - \sqrt{a})^2 y \leq ax + (1 - \sqrt{a})^2 y + 2\sqrt{ax(1 - \sqrt{a})y}$, what is always true, given that $2\sqrt{ax(1 - \sqrt{a})y} \geq 0$ always for the domain of definition of the function. Therefore, $f(x) \in K_{0.5}^1$.

Another not less interesting case is $f(x) = -\frac{1}{10000}x^2 + \frac{1}{100}x$ in any piece of the real domain $[0, 100]$. $f(x)$ is not convex either.

- K_s^2: Suppose it is true. Then $-\frac{1}{10000}(ax + (1 - a)y)^2 + \frac{1}{100}(ax + (1 - a)y) \leq -\frac{a^s}{10000}x^2 + \frac{a^s}{100}x - \frac{(1-a)^s}{10000}y^2 + \frac{(1-a)^s}{100}y$. Such implies $-a^s x^2 - (1 - a)^2 y^2 - 2ax(1 - a)y + 100ax + 100(1 - a)y \leq -a^s x^2 + 100a^s x - (1 - a)^s y^2 + 100y(1 - a)^s$. Therefore, $-(1 - a)^2 y^2 - 2ax(1 - a)y \leq -(1 - a)^s y^2$. This last inequality will lead us to two values for y: $y = 0$ and $-(1 - a)^2 y - 2ax(1 - a) + (1 - a)^s y = 0 \iff y = \frac{2ax(1-a)}{(1-a)^s - (1-a)^2}$. We are interested in the values which will make y

become non-positive. Therefore, once the parabola is downwards, we are interested in $y \leq 0$ or $y \geq \frac{2ax(1-a)}{(1-a)^s-(1-a)^2}$. With such restrictions in place, the inequality will always be verified. Therefore, $f(x) \in K_s^2$, any s, values of y restricted, in function of x.

- K_s^1: Suppose it is true. Then $-\frac{1}{10000}(ax+(1-a^s)^{\frac{1}{s}}y)^2+\frac{1}{100}(ax+(1-a^s)^{\frac{1}{s}}y) \leq -\frac{a^s}{10000}x^2+\frac{a^s}{100}x-\frac{(1-a^s)}{10000}y^2+\frac{(1-a^s)}{100}y$. Such implies $-a^2x^2-(1-a^s)^{\frac{2}{s}}y^2-2ax(1-a^s)^{\frac{1}{s}}y+100(ax+(1-a^s)^{\frac{1}{s}}y) \leq -a^sx^2+100a^sx-(1-a^s)y^2+100(1-a^s)y$. Therefore, $-(1-a^s)^{\frac{2}{s}}y^2-2ax(1-a^s)^{\frac{1}{s}}y+100(1-a^s)^{\frac{1}{s}}y \leq -(1-a^s)y^2+100(1-a^s)y$. This way we get two possible values for the roots of y: $y=0$ or $-(1-a^s)^{\frac{2}{s}}y-2ax(1-a^s)^{\frac{1}{s}}+(1-a^s)y=0 \iff y=\frac{2ax(1-a^s)^{\frac{1}{s}}}{(1-a^s)-(1-a^s)^{\frac{2}{s}}}$. Same reasoning as before leads us to $y \leq 0$ or $y \geq \frac{2ax(1-a^s)^{\frac{1}{s}}}{(1-a^s)-(1-a^s)^{\frac{2}{s}}}$. The beauty of all this is that the practical results always confirm previous deductions from Analysis. Notice that the bound for K_s^1 is lower than that for K_s^2, what makes s_2 fit in s_1, but not automatically vice-versa, very special conditions being necessary for the fitting process to take place.

Notice that the existence of a minimum interval, between x and y, is necessary to keep the definition of S-Convexity viable as well...Whilst, in Convexity, we work with same proportion, two straight lines, in S-Convexity we work either with exponential to the limit curve and straight line to domain, or with inverse of exponential in the domain and straight line 'reasoning' in the limit curve...

There is obvious a mismatch between the proportions, and the need to hold a minimum interval to allow the proportion to work. Before, with Convexity, we picked ax with $af(x)$. Now, we will pick either ax with $a^sf(x)$, or $a^{\frac{1}{s}}x$ with $af(x)$. This way, we need to add, to function domain, the piece corresponding to $(a^s-a)x$, or $(a^{\frac{1}{s}}-a)x$, at least.

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Definition 5. *A function $f: X->\Re$ is said to be s_1-convex if the inequality*

$$f(\lambda x+(1-\lambda^s)^{\frac{1}{s}}y) \leq \lambda^s f(x)+(1-\lambda^s)f(y)$$

holds $\forall\lambda \in [0,1]$*;* $\forall x < y \in X$*;* $X \subset \Re_+$.

Remark 4. If the complementary concept is verified, then f is said to be s_1-concave.

Definition 6. *A function* $f : X- > \Re$ *is said to be* s_2*-convex if the inequality*

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds $\forall\lambda \in [0,1]$*;* $\forall x < y \in X$*;* $X \subset \Re_+$.

Remark 5. If the complementary concept is verified, then f is said to be s_2-concave.

Notice, as well, that if we allow $x = y$ in s_2-Convexity, for instance, we also get a further restriction which is equivocated: $f(0) \geq 0$. These are all hints that such thing cannot, possibly, happen.

4 Limiting curve

The same way that the chord joining $f(x)$ to $f(y)$-corresponding to the verification of the Convexity property of the function f in the interval $[x,y]$-forms the limiting height for the curve representing f to be at, limit included, in case f is convex ([2]), and this chord is represented by $\lambda f(x) + (1-\lambda)f(y)$, all analytically expressed by the most usual statement of the property of Convexity, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$, there is a 'bent line' representing the limiting height for the curve f to be at, limit included, in case f is s_1-convex, $s_1 \neq 1$ ([4]). This curve is represented by $\lambda^s f(x) + (1-\lambda^s)f(y)$, for each chosen $s_1 = s$ between 0 and 1. $s = 1$, however, is taken away from our writings obviously because if $s = 1$ one gets the property of Convexity and the chord again.
It is not difficult to intuitively see that all other values of s, once $0 < s \leq 1$, are going to define limiting curves which approach the chord $\lambda f(x) + (1-\lambda)f(y)$ from above: the greater the value of s, the closer the limiting curve will be to becoming the mentioned chord.
The relevance of this limiting curve is extremely obvious: If we cannot precisely define it, there is no way we can classify a curve as $S-$convex by

means of our naked eyes (nor even if they are not naked!). On the other hand, in getting to know the details about such a limiting curve, such as maximum height, one finds ways of excluding curves that are not of that sort immediately. In understanding the exact shape of the limiting curve, one learns how to build Geometry tools to work out $S-$Convexity by hand. Easy to see how such study makes computational work easy as well.

K_s^1

From the definition, we know that this limiting curve is smooth (f is a C^1), grows from $f(x)$, up to the middle, and then decreases to the side of $f(y)$, all evenly, once $a^s f(x) = b^s f(x)$. With all this, we can be sure there is a maximum point in this curve and this maximum point should be in the middle of it, given the 'give and take' nature of the expression. As the sum of the terms in $L(s_1)$ will always be one, the maximum has to be 0.5 for each one of the parcels, so that $a = b = 0.5$ on its maximum moment, and the maximum point is actually $2^{1-\frac{1}{s}}$.

5 Effects of S on the limiting curve

Our limiting curve has a few aspects to it: maximum height, length, and local inclination. We try to deal with each one of them in a sequence:

- **Height:** Consider a bijection between the interval $[0,1]$ and the convex combination $\lambda f(x) + (1-\lambda)f(y)$, which is obvious if we consider that λ is between 0 and 1 (In one extreme, with $\lambda = 1$, one gets $f(x)$. In the other extreme, with $\lambda = 0$, one gets $f(y)$).

  For each λ and, therefore, each result $\lambda f(x) + (1-\lambda)f(y)$, it corresponds a member of the domain, $(\lambda x + (1-\lambda)y)$, whose image should be compared to that height and should prove itself to be less than, or equal to, it to allow the function to be classified as convex. One knows that $a^s f(x) + b^s f(y)$ is a continuous curve if f is continuous (f is always continuous according to our definition).

  When $a + b = 1$ is under consideration in convex functions, $af(x) + bf(y)$ naturally spans all the interval between x and y (as well as the interval between $f(x)$ and $f(y)$). On the other hand, a, as well as b, will definitely span $[0, 1]$. And a^s should be a continuous function, as well as b^s. A continuous function should take compact intervals into compact intervals. If both a^s and $f(x)$ are continuous, so is their multiplication and the sum with $b^s f(y)$.If both of them have images in compact intervals, their sum also does. With all that, we can safely state that $a^s f(x) + b^s f(y)$ will span the interval $[c, d]$, which contains $f(x)$ and $f(y)$. Because both a and b span $[0, 1]$ and a^s, b^s are 'twin' functions, the images should be exactly the same. It is then not hard to see that $a^s + b^s$ should give us a compact $[0, 2]$ as its range of action whenever $\{a, b\} \subset [0, 1]$. However, we limit the value to 1, in order to hold similar behavior to that presented by the Convexity line: $L(s_1)$ will then be close to the straight line, as close as it may be, still being an exponential. In the convex case, $a + b$, one can see that the image gets reduced to one, the result of that sum, via definition, being one. That results in a chord between $f(x)$ and $f(y)$, also via definition. In the exponential case, with a ratio as exponent, but with the sum limited to one by force, one will have same size as in Convexity. Once the peak of $a^s + b^s$ will occur when $a = b$, it is easy to see that $2a^s$ gives us the value of the maximum. $L(s_1)$ will, however, 'amputate' several points from $L(s_2)$ (all those having $a^s + b^s > 1$), therefore closing lower than $L(s_2)$. The rule does not change, what changes is points considered. Once the limitation is made clear via definition, obvious maximum height for $L(s_1)$ is one.

- **Local inclination:** The local inclination of our limiting curve is easily found by means of the first derivative of $f(\lambda) = \lambda^s f(x) + (1 - \lambda^s) f(y)$. Therefore, the inclination is $f'_s(\lambda) = s\lambda^{s-1} f(x) - s\lambda^{s-1} f(y)$ and varies accordingly to the value of f.

⊚

CLAIM 1 (see [2, PEARCE, p.283]): **An s_1-convex function is always non-decreasing in $(0, \infty)$, but not necessarily in $[0, \infty)$.**

Proof. So, we are basically to prove the non-veracity of CLAIM 1. Claim 1 consists of 2 parts (non-decreasing function=part 1; where it is non-decreasing=part 2). The non-veracity of Part 2 has already been dealt with in [9]. The first part has been put down in this very paper, earlier on. Not only we have provided the reader with an actual example of an s_1-function which is decreasing in the interval claimed to be impossible bearer of such, but we also have provided the reader with the reasons as to why such is untrue. □

The Lemma which follows should settle this matter for good:

Lemma 1. *In the definition of any sort of $S-$Convexity, it is found, as basic enthymeme, that one cannot, ever, possibly, hold $x = y$, for the soundness of their theory.*

Proof. In this very paper, we mention at least one example of an s_1-convex function, which is also decreasing. Therefore, such statement cannot, ever, be proven true. We now hold an essential problem with a proof we, ourselves, got confused about, and even claimed to have refereed, at some stage, that of Dragomir et al. regarding the fact mentioned also at [3]. Even though the counter-example proves the fact, we do need to find a fallacy with the proof, which is analytical. Basically, we work there with approximations, fact disregarded by Dragomir et al. in their report of the proof. In being S-Convexity a majorly geometric definition, as much as Convexity is, it is fundamental to hold at least two points in the reals, and such has to be accompanied by a multitude of them (in the reals universe), therefore making it impossible for x to 'copy' y with perfection (or for use to sustain that it is possible to accept $x = y$ as possible values to be applied to the definition). The proof is also of doubtful nature if made to check on consistency of the Convexity definition, for instance. However, we explain the fact via continuity and irrelevance of figures coming after the decimal mark, making the values 'the same'.
We hold several options to go about the proof of the above Lemma:

- **Point of proof 1**: Even for Convexity, making $x = y$ in the definition statement, seems to be analytically unsound. Notice that we hold two coefficients for the domain points. One 'takes' what the other 'puts in', basically, making use of 1 size as basis. The major question to be asked then is whether we could take so little from each extreme at a point of making them be the same, once no mathematical formulae would be well-posed if allowing for x, y to actually mean only x. There is obviously assumption of 'necessity', or 'imperative of force', there. This is the explanation, or justification, via well-posedness theory;
- **Point of proof 2**: There is 1, which appears there as figure to measure distance between points of the domain picked by formulae as basis. The fact obvious knocks down any trial of making x converge to y, or vice-versa. Basically, not even in Convexity, should one make use of such a reasoning. It all sounds equivocated. That is the distance between domain points being used as argumentation;
- **Point of proof 3**: There seems to be 'algebraic' allowance for a person to assume $x = y$ in the domain point, which is supposed to actually mean point between two other points (x and y in the formulae), that is, it seems 'algebraically' sound to do so. When writing $x = y$ in the formulae, we actually notice that, for any value of a picked, only one of the variables will remain inside of brackets, validating that reasoning. However, analytically, one cannot think of such. The analytical definition is matched to a geometric definition, which is clear as to the necessity of an 'interval', which is non-degenerated, in which to measure a function as to its pertinence to the S-Convexity group. No inconsistencies can be allowed in Analysis. Dragomir et al. seems to change distance 1 into distance 0 between two at least psycho-linguistically implied-to-be different points of the domain of the function. However, 1 has to do with same line of reasoning as that of proportion (or 'scaling'). Can one propose

a 0 unit factor for scaling? Do not think so...;

– **Point of proof 4**: The consistency of Mathematics guarantees that x must be fully different from y. S-Convexity is supposed to be an extension of Convexity, not mattering its sense. Any extension must guarantee inclusion of whatever is being extended...they both include Convexity algebraically. Allowing $x = y$ in their formulae, however, makes of every convex function a non-decreasing function with $f(0) \geq 0$, what is absurd. Another obvious thing is that, for s_1, not even algebraically possible such is, for there is no possible value for a in that situation.

Remark 6. Interesting enough is noticing that one should actually find x and x in each definition of each type of S-Convexity, therefore in the own basic definition for Convexity, as well. Making use of two variables, instead of one, for a definition which is supposed to live in the real domain of one dimension, is obviously mathematically unsound. Such misconception cannot be allowed at all in Science. Science should hold most objective language as possible, meaning that whatever adds confusion to any definition should not be there. The way the definition is presently found, one does not understand, as Dr Sever and his group did not, that the concept is not living in $\Re^2$, because it actually is living there algebraically, even though that is not the case with the geometric definition...Such fact is not an excuse, however, for a whole editorial board to accept atrocities and publish that, exposing millions of researchers to possible exponential growth of mistake, for it is obviously the case that an equivocated definition, that is, in breach of rules of well-posedness, may generate as much confusion as a law which is badly written. Such has to be fixed, and we will endeavour doing it here, hoping nobody 'progresses' further in 'digression' of Science for misunderstanding of its foundational concepts...Our 'ultimate fixing' follows the proof of the Lemma which will, once more, reassure the reader that it is impossible to accept repeated values in the construction of the domain point picked by the left side of the inequality of definition of any type of S-Convexity, of which Convexity is a special case ($s = 1$).

Following any of the lines of reasoning above will easily justify the assertion contained in our Lemma. $\square$

In the fixing of the definitions of S-convexity, we also add the following Theorem, with proof in [7]:

Theorem 2. *For every function in K_s^2, it has to be the case that the whole set of images is either located entirely on the positive share of the counter-domain, or entirely on the negative share of it. In the case of the reals, $(f(x) \in \Re_+)\underline{\vee}(f(x) \in \Re_-), \forall x \in D_f$.*

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Definition 7. *A function $f : X- > \Re$ is said to be s_1-convex if the inequality*

$$f(\lambda x + (1-\lambda^s)^{\frac{1}{s}}(x+\delta)) \leq \lambda^s f(x) + (1-\lambda^s)f(x+\delta)$$

holds $\forall \lambda \in [0,1]$; $\forall x \in X$; $X \subset \Re_+$, for all fixed $\delta \in \Re_+^*, \delta \geq (0.5 - 0.5^{\frac{1}{s}})$.

Remark 7. If the complementary concept is verified, then f is said to be s_1-concave.

Definition 8. *A function $f : X- > \Re_+ \underline{\vee} \Re_-$ is said to be s_2-convex if the inequality*

$$f(\lambda x + (1-\lambda)f(x+\delta)) \leq \lambda^s f(x) + (1-\lambda)^s f(x+\delta)$$

holds $\forall \lambda \in [0,1]$; $\forall x \in X$; $X \subset \Re_+$, for all fixed $\delta \in \Re_+^*, \delta \geq (0.5^s - 0.5)$.

Remark 8. If the complementary concept is verified, then f is said to be s_2-concave.

Remark 9. In being Convexity a special case of either of the definitions above, or of both, we do not need to ever use the definition of convex function anymore in the literature. We may simply replace it with its extension, and work with them always as a family of functions, rather than a split group.

6 On inconsistency of denominations, or referents, or sigmatoids, in Science: S-Convexity

Apparently, as seen on [10], a group of researchers from Statistics have missed out not doing enough survey, and came up with new definition for an old mathematical expression. Unfortunately, that generates inconsistency in Science, and the ones to fix denomination have to be those with most recent definitions. In quick consultation to [11], one finds out that, with I. Prof. M. Denuit, the name $S-$Convexity appears by 1997. That means re-defining the mathematical concept which became better known in the scientific media by Hudzik and Maligranda's paper, concept created well before that decade (Hudzik's paper dates from 1994): Such is unethical. We do not hold inspecting bodies for ethics in Mathematics, or Statistics, what makes our progress, in those pieces of Science, something very unlikely to happen with consistency. However, common sense would definitely lead Dr. Denuit, and fellows, to modify the symbol currently used to designate their concept, which differs substantially from the one of equal name in Mathematics. Basically, in their work, S-Convexity relates to stochastic processes, majorly. The value of S there differs substantially from the mathematical value. On top of that, it has to do with time series and, therefore, evolution in time of something (like the stock market). The properties of his defined processes are different, in nature, from those of the mathematical concept. In Statistics, actually making more sense than in Mathematics, they apparently have associated the name 'S-Convexity' to symmetric distribution: S from <u>s</u>ymmetric. However, in Mathematics, square root of a real, for example, is an S-convex function with no symmetry whatsoever to it: The concepts are clearly incompatible. On top, the word 'ordering' is used associated with the concept S-Convexity in the works of Denuit et al.: Ordering is not involved in the mathematical concept (In Mathematics, perhaps, at most, ordering would have to do with the families' members evolution (each specific value for s_1, for instance)).

To make it short, Dr. Denuit et al. must add something extra to their current symbols (and denominations), to make them differ from those in Mathematics, or find a way of connecting both concepts, taking Hudzik's definition as a basis for such.

7 Conclusions

In this paper, we have provided the reader with as many geometric details as possible on the limiting curve for s_1-Convexity, now equating its use to that of Convexity, which is just a particular case of s_1-Convexity, occurring when $s_1 = 1$. Our work provides any reader with sufficient tools to geometrically classify a curve as s_1-convex, or not, and even determine the value of s_1 by hand.
As a side extraordinary result, we get to fix the definition of Convexity, to a palatable definition, in terms of Real Analysis.
Results observed, we seem to have completed the foundational share for $S-$Convexity, in the Cartesian universe. Future work shall bring more on Jensen's inequality (continuous versions), as well as on Complex Analysis.

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