Notes on one of the generators of the $S-$convexity phenomenon

I.M.R. Pinheiro

PO BOX 12396, A'Beckett st, Melbourne, Victoria, 8006, AUSTRALIA

e-mail: $mrpprofessional@yahoo.com$

Abstract: In this one more precursor paper, we deal with one specific model which is currently found in the literature on $S-$convex functions: polynomial. We not only review this model, making it look more mathematically solid, but we also extend it a bit further. On top of that, side remarks of quality are made regarding the subject.

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1. Introduction

In this paper, we deal with a few issues related to models proclaimed to generate $S-$convex functions, that is, with theorems, which are already found in the literature regarding that. In the lines which follow, we investigate the polynomial model in depth, criticizing it and fixing it, as well as extending it.

Generators

U. R[1]*.: Related problems found in the literature*

U. R.: PROBLEM 1) Basic model: how good, or adequate, it actually is?

Regarding polynomial models, what is found in the literature so far is:

From Dragomir and Pierce [1], page 283:

M) For $0 < s < 1$, $\{a, b, c\} \in \Re$, $u \in \Re_+$, take M to be:

$$f(u) = a, \quad u = 0$$

and

$$f(u) = bu^s + c, \quad u > 0.$$

U. R.: PROBLEM 2) Conditions of fitting found in the literature so far: Are they correct?

1.B) ***Generators for*** s_1-***functions***

1.B.1) If, in the model just mentioned, $b \geq 0$ and $c \leq a$, then $f \in K_s^1$, which we shall name A, that is, A will stand for set of conditions for which the functional model M is a generator of examples of functions in K_s^1.

1.C) ***Generators for*** s_2-***functions***

1.C.1) In Dragomir and Pierce [1], p. 292, we read that if $b > 0$ and $c < 0$ then $f \notin K_s^2$;

1.C.2) The same source, same page, also reveals that if $b \geq 0$ and $0 \leq c \leq a$ then $f \in K_s^2$.

U. R.: PROBLEM 3) From the same source, [1], p. 283, we read that if A is found but $c < a$, that is, not allowing $c = a$, then we have a non-decreasing

[1]Our acronym for 'under review'.

function in $(0, \infty]$ but not necessarily in $[0, \infty]$. This is a severely odd remark, once there is mathematical proof, also provided in Dragomir and Pierce [1], as to state that any function in K_s^1 will fall into the non-decreasing category for the $(0, \infty]$ case. Problem 3 has already been addressed by us in [3]. Solved.

In the sections that follow, we deal with:

- Terminology;
- Definitions;
- A simple extension, in the polynomial case, for M, as well as A, also addressing of problems encountered;
- Conclusion.

2. Terminology

We use the same symbols and definitions presented in Pinheiro [2]:

- K_s^1 for the class of S-convex functions in the first sense, some S;
- K_s^2 for the class of S-convex functions in the second sense, some S;
- K_0 for the class of convex functions;
- s_1 for the variable S, $0 < S \leq 1$, used in the first definition of S-convexity;
- s_2 for the variable S, $0 < S \leq 1$, used in the second definition of S-convexity.

Remark 1. The class of 1-convex functions is just a restriction of the class of convex functions, that is, when $X = \Re_+$,

$$K_1^1 \equiv K_1^2 \equiv K_0.$$

3. Definitions so far

Definition 3. *A function $f : X- > \Re$, $f \in C^1$, is said to be s_1-convex if the inequality*

$$f(\lambda x + (1-\lambda^s)^{\frac{1}{s}} y) \leq \lambda^s f(x) + (1-\lambda^s) f(y)$$

holds $\forall \lambda \in [0,1]$, $\forall x, y \in X$ such that $X \subset \Re_+$.

Definition 4. *f is called s_2-convex, $s \neq 1$, if the graph lies below a 'bent chord' (L) between any two points, that is, for every compact interval $J \subset I$, with boundary ∂J, it is true that*

$$\sup_J (L - f) \geq \sup_{\partial J} (L - f).$$

Definition 5. *A function $f : X- > \Re$, in C^1, is said to be s_2-convex if the inequality*

$$f(\lambda x + (1-\lambda) y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds $\forall \lambda \in [0,1]$, $\forall x, y \in X$ such that $X \subset \Re_+$.

4. New Definition

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Definition 6. *A function $f : X- > \Re$, $f \in C^1$, is said to be s_1-convex if the inequality*

$$f(\lambda x + (1-\lambda^s)^{\frac{1}{s}} y) \leq \lambda^s f(x) + (1-\lambda^s) f(y)$$

holds $\forall \lambda \in [0,1]$, $\forall x < y \in X$ *such that* $X \subset \Re_+$.

3. Simple extension, polynomial case model, and addressing of the problems

From here onwards, each time we come up with a new result, we place a ★ close to it to indicate it is such, and each time we prove old statements, published in the literature, so far, are equivocated, we place a ⊚ close to them to indicate it is such.

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Theorem 6.1. *K_s^1 is not a class formed by exclusively non-decreasing functions.*

Proof. There is obviously no sense in using the same member of the domain twice in one essentially geometric definition, always referring to limit curves, like in convexity. However, one could use that reasoning as a draft for work with limits. Such refinement will keep their results, in terms of the K_s^1 group being formed by non-decreasing functions. Suffices, then, considering the second

point as close to the first as wanted, so that there is no essential difference, given continuity, between the second and the first point considered.
We actually kill any chances of such claim by Dragomir et al. being true with the following Lemma:

Lemma 1. *In the definition of any sort of $S-$convexity, it is found, as basic enthymeme, that one cannot, ever, possibly, hold $x = y$, for the soundness of their theory.*

Proof. Easy examples of s_1-convex functions, which are also decreasing, are found (Take, for instance, $f(x) = -\frac{1}{10000}x^2 + \frac{1}{100}x$ in $y \geq \frac{2ax(1-a^s)^{\frac{1}{s}}}{(1-a^s)-(1-a^s)^{\frac{2}{s}}}$). Therefore, such a statement cannot, ever, be proven true, for validity of own Mathematics. We now hold an essential problem with a proof we, ourselves, got confused about, and even claimed to have refereed, at some stage, proof of Dragomir et al., which we actually made equivocated use of at $[6, PINHEIRO]$. Even though the counter-example proves the fact, we do need to find a fallacy with the proof, which is analytical. Basically, we work there with approximations, fact disregarded by Dragomir et al. in their report of the proof. In being S-convexity a majorly geometric definition, as much as convexity is, it is fundamental to hold at least two points in the reals, and such has to be accompanied by a multitude of them, therefore making it impossible to accept $x = y$ as a possibility. The proof is also of doubtful nature if made to check on consistency of the convexity definition, for instance. However, we explain the fact via continuity and irrelevance of figures coming after the decimal mark, making the values 'the same'.
We hold several options to go about the proof of the above Lemma:

- Point of proof 1: Even for convexity, making $x = y$ in the definition statement, seems to be analytically unsound. Notice that we hold two coefficients for the domain points: One 'takes' what the other 'puts in', basically, making use of 1 size as basis. The major question to be asked then is whether we could take so little from each extreme, at a point of making them being the same, once no mathematical formulae would be well-posed if using x, y to mean only x. There is obviously assumption of 'necessity', or 'imperative of force', there. This is the explanation, or justification, via well-posedness theory;

- Point of proof 2: There is 1, which appears there as figure to measure distance between points of the domain picked by formulae as basis. The fact obvious knocks down any trial of making x converge to y, or vice-versa. Basically, not even in Convexity, should one make use of such a reasoning. It all sounds equivocated. That is the distance between domain points being used as argumentation;

- Point of proof 3: There seems to be 'algebraic' allowance for a person to assume $x = y$ in the domain point, which is supposed to actually mean point between two other points (x and y in the formulae), that is, it seems 'algebraically' sound to do so. When writing $x = y$ in the formulae, we actually notice that, for any value of a picked, only one of the variables will remain inside of brackets, validating that reasoning. However, analytically, one cannot think of such. The analytical definition is matched to a geometric definition, which is clear as to the necessity of an 'interval', which is non-degenerated, in which to measure a function as to its pertinence to the S-convexity group. No inconsistencies can be allowed in Analysis. Dragomir et al. seems to change distance 1 into distance 0 between two supposedly different points of the domain of the function. However, 1 has to do with same line of reasoning as that of proportion, or 'scaling'. Can one propose a 0 unit factor for scaling? Do not think so...;

- Point of proof 4: The consistency of Mathematics guarantees that x must be fully different from y. S-convexity is supposed to be an extension of convexity, not mattering its sense. Any extension must guarantee inclusion of whatever is being extended...they both include Convexity algebraically. Allowing $x = y$ in their formulae, however, makes of every convex function a non-decreasing function with $f(0) \geq 0$, what is absurd. Another obvious thing is that, for s_1, not even algebraically possible such is, for there is no possible value for a in that situation.

Following any of the lines of reasoning above will easily justify the assertion contained in our Lemma. With this Lemma, our assertion is trivially proven, once Dragomir et al. makes their result out of the 'taken-for-granted' assumption that x may be equal to y in the definition of the first sense of convexity. □

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Lemma 2. *The sum of two $S-$convex functions is also an $S-$convex function.*

Proof. Take f and g to be $S-$convex (f and g inside of the same class: either K_s^1 or K_s^2). We now have: $f(ax+by)+g(ax+by) \leq a^s(f+g)(x)+b^s(f+g)(y)$. This implies $(f+g)(ax+by) \leq a^s(f+g)(x)+b^s(f+g)(y)$, that is, the sum of two $S-$convex functions is, indeed, an $S-$convex function, as wished for. □

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Theorem 6.2. *Every s_1-convex function is also s_2-convex.*

Proof. Proving the statement is trivial: Suppose it is not. We then will find a such that the s_1 property is verified, but s_2 is not. With this, there is an element of the domain, situated between x and y chosen for both cases, which is higher in image than $a^s f(x) + (1-a)^s f(y)$. The same element is found in s_1 in the situation of being less than $a^s f(x) + (1 - a^s)f(y)$. Notice that this implies such element, call it y, being in an impossible to be situation, according to our previous proofs $(a^s f(x) + (1-a)^s f(y) < f(y) \leq a^s f(x) + (1-a^s)f(y))$. This can never happen due to the fact, proven before, that $(1-a^s) \leq (1-a)^s$. Therefore, $K_s^1 \subset K_s^2$. The converse reasoning does not yield any conflict. □

□

6.1 Past statements reviewed

- If it suffices that each member of the sum is $S-$convex, why would it be necessary to hold the restrictions $b \geq 0$ and $c \leq a$, for one of the cases, and $\{a, b, c\} \in \Re$, or $u \in \Re_+$?

- The current literature seems not to account for the situations in which the domain member is zero. Taking it to be zero will lead us, via approximation reasoning, to $f(0) \leq (a^s + b^s)f(0)$. That means $(1-(a^s+b^s))f(0) \leq 0$. If in K_s^1, that will always be true. In the second sense, however, that would bring us to a negative value multiplied by $f(0)$ being non-positive. What that means is that $f(0) \geq 0$ is a mandatory condition for a function to belong to K_s^2, $s \neq 1$. We have dealt with this issue already, even in this very paper, earlier on.

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Claim 1.B.1 is not true. The fact that $b \geq 0$ and $c \leq a$, in M, will not guarantee that f belongs to K_s^1,that is, A should be revisited and revised, in all. Furthermore, if $c \neq a$, in M, we do not hold continuity. We have already made this point to exhaustion. Basically, we do wish for a continuous function, it cannot make any sense to have zero apart from rest of the domain. Such equivocated assertion may only have emerged from the same imagination scenery in which the function would be non-decreasing for non-negative domain, without zero: It is all nonsense...We believe to have already addressed this issue. Continuity does imply closed interval in this case, and the own definition does demand continuity, cannot go without it. The major intention in convexity extensions is doing similar geometric job, which is basically spanning all space of images between the images of the extremes of the domain interval chosen. Therefore, continuity is implied by the definition, at least in the interval under consideration.

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Theorem 6.3. *For every function in K_s^2, it has to be the case that the whole set of images is either located entirely on the positive share of the counter-domain, or entirely on the negative share of it. In the case of the reals,* $(f(x) \in \Re_+)\underline{\vee}(f(x) \in \Re_-), \forall x \in D_f$.

Proof. From [5], we see that the derivative of the limit curve for K_s^2 demands the whole function to belong to either the positive side of the real axis entirely, or to its negative side, entirely, with no mix allowed. This because if $f(x)$ is found on the positive side, but $f(y)$ is found on the negative side of the axis, one is left with no possible value for λ in terms of a maximum existing there, and such is not acceptable, or compatible, with our deductions and definitions. □

Remark 2. Notice that the above theorem would make of K_s^2 a proper extension for convexity in particular cases of occurrence of its phenomenon only (where the counter-domain restriction occurs originally).

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Lemma 3. $c\left(\frac{u}{n}\right)^s + d$, *where* $\{u, d\} \in \Re_+^n$, *is an* $S-$*convex function.*

Proof. Notice that d is irrelevant because one ends up with $(a^s + b^s)$ its value to the right of the inequality, which is trivially bigger than, or equal to, itself, in our case. For matters of proof, we know that as long as c is non-negative, c is then made also irrelevant. Notice, also, that the null function will always satisfy the inequality for $S-$convex functions. Consider then analyzing $\left(\frac{ax+by}{n}\right)^s \leq a^s\left(\frac{x}{n}\right)^s + b^s\left(\frac{y}{n}\right)^s$, as to its veracity. It looks trivial to infer the inequality is correct (raise to the inverse of the power, both sides, for instance). The result will be verified the same way for several dimensions (see [5], for instance, for definitions). □

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Theorem 7. *A model for* $S-$*convex functions, of either sense, is* S*:*

$$f(u) = a_1 u^{s_1} + a_2\left(\frac{u}{n}\right)^{s_2} + a_3\left(\frac{u}{n-1}\right)^{s_3} + \ldots + a_m\left(\frac{u}{n-m}\right)^{s_m} + a_{m+1},$$

$u \neq 0, 0 \leq m < n, \{m, n\} \subset N, 0 < s_n \leq 1, (a_n) \subset \Re_+,$ $\{u, a_{m+1}\} \in \Re_+^x, f$ *in* $K_s^1 \bigcup K_s^2$; $(f(x) \in \Re_+) \underline{\vee} (f(x) \in \Re_-), \forall x \in D_f$, f *in* K_s^2.

Model S models a function which is s_k-convex, where $k = \min\{s_1, ..., s_m\}$.

Proof. Lemmas plus all previously mentioned theorems entitle us to think the theorem is accurate. □

Remark 3. Notice that we do like continuous functions better, and see no sense in defining $f(0)$ in a way of making our S-convex function discontinuous. On top of that, we hold a geometric definition for the concept of S-convexity; a clear one. That definition implies continuity of the domain, as well as in the image, so that we cannot really afford having a discontinuity there.

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On the chance that it is possible, or necessary, to **define** $f(0)$ **as something apart** from the rest of the S-convex function, in terms of model:
It makes no sense at all thinking of a discontinuous S-convex, or convex, function. Perhaps a set, but not a function...Basically, the definitions are assembled in a way to imply continuity both in the domain piece under analysis and in the image piece resulting of the application of the function over that piece of domain.

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Regarding **1.C.1**, $b > 0$ and $c < 0$ then $f \notin K_s^2$, the assertion is trivially not true. Take, for instance, the function:

$$f(0) = a_0$$

$$f(u) = a_1 u^s + a_{m+1},$$

where $u^s \geq 1$ and $a_1 > 0$. We then have $f(ax + by) = a_1(ax + by)^s + a_{m+1}$. On the other hand, $a^s f(x) = a^s a_1 x^s + a^s a_{m+1}$ and $b^s f(y) = b^s a_1 y^s + b^s a_{m+1}$. Trivially, not mattering the value of the independent term, it is not relevant for our judgment on $S-$convexity pertinence or not (if it is in K_s^1, elimination is direct on inequality formation; $a_{m+1} \geq 0$ guarantees the pertinence to K_s^2, but that does not imply it to be only case. If a_{m+1} is close to zero, but less than zero, the function of this shape should still be located in K_s^2 provided modulus of the term with the variable overcomes its value), regarding the function.
As one of the infinitely many possible examples, take $f(x) = 2x^{0.5} - 0.0001$. See: $f(\lambda x + (1-\lambda)y) = 2(\lambda x + (1-\lambda)y)^{0.5} - 0.0001 \implies 2(\lambda x + (1-\lambda)y)^{0.5} - 0.0001 \leq \lambda^{0.5} 2x^{0.5} - \lambda^{0.5} 0.0001 + (1-\lambda)^{0.5} 2y^{0.5} - (1-\lambda)^{0.5} 0.0001$. By raising both sides to the second power, the inequality is trivially verified.

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The second assertion, in **1.C.2**, $b \geq 0$ and $0 \leq c \leq a$ then $f \in K_s^2$, is also not accurate. Notice that $b \geq 0$ should be OK, needed condition, $c \geq 0$ as well, but there is apparently no connection between $f(0)$ and rest, so that $c \leq a$ does not make sense there.

Remark 4. Paying attention to the mentioned page, in the source, $a = a_0$, $c = a_{m+1}$, and $b = a_1$. What is noticed then is that if discontinuity is verified, one would have $a \neq c$, making odd assertion true in almost full content, missing replacing $c < a$ with $c \neq a$.

8 On the name: What is special about the S?

Basically, a very well published researcher asked us this question. In his mind, there should be a special reason for the S to be there...However, a bit of research on the subject, easily found condensed at [6], will prove the name has originated on a very logical mind: both $r-$ and $R-$ convexity had already been defined by time of invention of the $S-$convexity! (one of the citations there refers to 1972). Hudzik's article is from 1994, Breckner's, from 1978...Therefore, it is simply logical that next sort of convexity would be called $S-$convexity...

As we have seen here, there is nothing to any of the functions which could hold any sort of connection with the letter, in special. In Statistics, they apparently have used it to indicate symmetry. There is a symmetry on the limit curve for S-convexity, so that this might be useful in a lecture, for instance, making it easy for the student to remember what S-convexity means.

As for our research power, this was all which was found to be true regarding its choice for baptism in being born.

6. Conclusion

In this paper, we review, and fix, the basic model for S-convex functions, as well as extend it inside of the polynomial scope. As a side result, we also review and fix assertions, which exist in the literature, about a few properties of the S-convex functions.

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