Notes on one of the generators of the s_2-convexity phenomenon III and insights on the application of our previous geometric results

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Abstract: In this one more paper, we try to create new models for the S-convexity phenomenon. As a side result, we nullify three previous results connected to the concept.

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1. Introduction

So far, we seem to be stuck with the polynomial model, when it comes to find examples of $S-$convex functions. However, in this paper, we will dare putting down this paradigm and provide the reader with trigonometric models as well. In the sections that follow, we deal with:

- Terminology;
- Definitions;
- Simple and basic trigonometric examples, which may be combined;

- Review of past results;

- Conclusion.

2. Terminology

We use the same symbols and definitions presented in Pinheiro [2]:

- K_s^1 for the class of S-convex functions in the first sense, some S;

- K_s^2 for the class of S-convex functions in the second sense, some S;

- K_0 for the class of convex functions;

- s_1 for the variable S, $0 < S \leq 1$, used in the first definition of S-convexity;

- s_2 for the variable S, $0 < S \leq 1$, used in the second definition of S-convexity.

Remark 1. The class of 1-convex functions is just a restriction of the class of convex functions, that is, when $X = \Re_+$,

$$K_1^1 \equiv K_1^2 \equiv K_0.$$

3. Definitions

Definition 3. *A function* $f : X -> \Re$, $f \in C^1$, *is said to be* s_1*-convex if the inequality*

$$f(\lambda x + (1 - \lambda^s)^{\frac{1}{s}} y) \leq \lambda^s f(x) + (1 - \lambda^s) f(y)$$

holds $\forall \lambda \in [0, 1]$, $\forall x, y \in X$ *such that* $X \subset \Re_+$.

Definition 4. f *is called* s_2*-convex,* $s \neq 1$, *if the graph lies below a 'bent chord' (L) between any two points, that is, for every compact interval* $J \subset I$, *with boundary* ∂J, *it is true that*

$$\sup_J (L - f) \geq \sup_{\partial J} (L - f).$$

Definition 5. *A function* $f : X -> \Re$, *in* C^1, *is said to be* s_2*-convex if the inequality*

$$f(\lambda x + (1 - \lambda) y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y)$$

holds $\forall \lambda \in [0, 1]$, $\forall x, y \in X$ *such that* $X \subset \Re_+$.

4. Simple new model, trigonometric

From past work, we will use one conclusion, one definition (geometric), and a theorem (from [3], [2], and [4]):

Conclusion:

2^{1-s} is always the maximum height for our curve.

Theorem 5.1. *The sum of two $S-$convex functions is also an $S-$convex function.*

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Lemma 1. *If a function is represented graphically by a smooth curve, continuous, and the distance between the straight line formed between any couple of points in the curve, and the actual function curve is at most one, excluding one from consideration, as seen in* [3]*, we then have an s_2-convex curve.*

Proof. Such is obvious and follows directly from the definition combined with our explanations and deductions proven in [3]. □

We now consider a few trigonometric functions on the grounds of their suitability for being mathematical models of $S-$convex functions.

- $\sin(x)$ and $\cos(x)$.

  Notice that it is redundant analyzing both functions here. Their graphical behavior, which is the only thing that matters for $S-$convexity, is practically the same, only different regarding time of occurrence. Therefore, whatever we deduce regarding one of them may be applied directly to the other, with subtle adaptation to domain intervals (ninety degrees).

The highest pitch of the sine function is achieved at one. With this, we already know that it is not possible to consider the first quadrant, as a whole, for our example in K_s^2, for instance, once 2^{1-s} will never reach the value two, only go close (given the limitations imposed upon the value of s). However, suffices taking a slightly shorter interval in the domain and our desired example is found. Say we then pick $[0, \frac{\Pi}{2})$, once that will give us what we need or even maximum fit, $[0, \Pi)$. Our distance then, regarding the possible straight line between those points and the actual curve will be app. 1.

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Theorem 6. $\sin(x)$ *is a model for* K_s^2*, if the domain is restricted to* $[0, \frac{\Pi}{2})$*, or (exclusively)* $(\frac{\Pi}{2}, \Pi]$*, or (exclusively)* $[0, \Pi)$*, or (exclusively)* $(\frac{3\Pi}{2}, 2\Pi]$*, as well as all other colinear angles*[1]*.*

Proof. As above. □

Corollary 1. $\cos(x)$ *is a model for* K_s^2*, if the domain is restricted to* $(0, \frac{\Pi}{2}]$*, or (exclusively)* $[\frac{\Pi}{2}, \Pi)$*, or (exclusively)* $(\frac{\Pi}{2}, \frac{3\Pi}{2}]$*, or (exclusively)* $[\Pi, \frac{3\Pi}{2})$*, or (exclusively)* $[\frac{3\Pi}{2}, 2\Pi]$.

5. Review of past works

★★★★Nullification of three past results★★★★

From [5] we read, on page 688:

[1]Remember that when we write one open end and another closed, in our intervals here, it could as well be the opposite in side placement, result being still adequate.

Theorem 6.1. *Let f be a function on $[a, b]$ which is s-convex in the second sense. Then for $a < y < z < b$ we have*

$$|f(y) - f(z)| \leq (z - y)^s \quad \max(f(b)/(b - y)^s, f(a)/(z - a)^s)$$

so the f is locally Hölder continuous of orders s on (a, b). Thus f is Riemann integrable on $[a, b]$.

There is a major problem, once more, with the proof of such theorem, as appears in [5]. Basically, one cannot first create a new variable, call it t, with the 'attrib' command from Maple, and define it based on other three already existing variables, as for the own theorem start, then make use of one of the variables, in which the creation of t was based, in the same expression as t, instead of expressing z as depending on same variables as rest of the expression. However, such step could represent minor typos, or distractions, and the proof still be accurate. So, first of all, re-writing the first inequality which appears on that proof, we get: $f((1-t)y+tb) \leq (1-t)^s f(y) + t^s f(b) \leq f(y) + t^s f(b)$. Given we impose K_s^2 pertinence to f, such would be correct, in principle. Next step, re-written, would be: $f((1-t)y+tb) - f(y) \leq t^s f(b)$, which still makes sense. However, if ever willing to call whatever comes in brackets, z, one would also have to change y into its expression depending on z, for that is for the best of Mathematics. On top of that, b would also appear expressed in terms of z. Such will actually invalidate the conclusion there present.

It is obviously worthless continuing on that line of thought. However, there is a major absurdity, not to say insanity, in the mentioned theorem (actually more than one): Hölder continuity refers to a constant multiplied by the difference between the variables mentioned to the left of the inequality...not to a function

of the variables instead...Please, if in doubt, check sources such as [1]...
It also does not make sense, even if that step were not there (no constants to right), writing that that implies Riemann integrability in a theorem, this is simply inadequate.
So, basically, that was never a theorem and should simply be erased from mathematical history, in full, as if it had never been published, refereed by so many people, from a whole editorial board...for the sake of ethics, we should simply forget it.

Still from [5] we read, on page 688:

Theorem 6.2. *Let f be a function on $[a, b]$ which is s-convex in the second sense. If $f(c) = 0$ for some $c \in [a, b]$ then $f(x) \leq f(y)$ if $c \leq x \leq y \leq b$ and $f(x) \geq f(y)$ if $a \leq x \leq y \leq c$.*

Well, from our past work and deductions, the theorem above is absurd. However, we also look into the details of the proof, not to say we do not hold consideration and respect for someone else's published work. Science must be about argumentation and who is technically sound.
So, there we go: The first line of the proof states that 'If $c \leq x \leq y \leq b$ then $f(x) \leq t^s f(c) + (1-t)^s f(y)$ if $x = tc + (1-t)y$. We may stop here, for the sake of all of us only reading the same remark twice. Once more, x, the variable, is defined based on other two and, in the definition of x, c, t, and y are all included. However, in the inequality formed, x appears as if it is independent from whatever appears to the right side...
If x were adequately mentioned in the inequality, however, the conclusion would not follow, what follows is something else. It is obvious that the last

inequality attained in the 'proof' would bring x in both sides, if Mathematics were ever considered...

6. Conclusion

In this paper, we present a new model for $S-$convex functions and, by presenting it, we believe to have exhausted the issues regarding finding examples for functions in K_s^2 for any interested researcher, once it will suffice repeat our procedures here, with the graph of functions, to determine, same way we do with convexity, whether a certain function belongs, or not, to K_s^2 (only). On the top of that, we review and nullify two previous results by Dragomir et al. ([5]).

7. Bibliography

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