$H - H$ Inequality for S-convex Functions

I.M.R. Pinheiro

PO BOX 12396, A'Beckett st, Melbourne, Victoria, 8006, AUSTRALIA

e-mail: $mrpprofessional@yahoo.com$

Abstract: In this one more paper on $S-$convexity, we referee, on the top of everyone else, the lower bound, proposed by S. Fitzpatrick et al., for K_s^2, by applying our own work to their earlier deductions. On the top of that, we define $H - H$ results also for K_s^1 and write a bit about why S_1-convexity is not a proper extension of the concept of convexity in what regards the set of the real numbers, but S_2 is. And, as a side result, we get to improve the range of choices for the Hermite-Hadamard inequality bounds, in terms of constants.
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1. Introduction

We now will try to produce an equivalent to the HH inequality for $S-$convex functions in general. On the way, we produce some quite nice results and revisions on even the own Hermite-Hadamard Inequality, so famous in the literature. In the sections that follow, we deal with:

- Hermite-Hadamard Inequality itself;
- Terminology;
- Definitions;
- Survey and remarks on a few old findings regarding HH and $S-$convexity as well as extension of results from convexity to $S-$convexity;
- Conclusion.

2. Hermite-Hadamard Inequality

The Hermite-Hadamard Inequality is mentioned, for instance, in Niculescu [1], as well as stated. It states that:

Theorem 2.1. *For $f : [a, b] -> \Re$, f being convex, it is always true that*

$$\text{(In. 0)} \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}.$$

Therefore, the so famous Hermite-Hadamard inequality is about the boundaries of the interval of the domain of the function and the effect of the midpoint domain point image as well as the effect of the images of the endpoints of the interval over the average weight of the area under the whole curve, of the whole function, over each individual element of the domain.
The evidence that this is true is there mentioned via approximation to the value of the integral via trapezoid rule and midpoint term.

Remark 1. The trapezoid rule for approximation of integrals actually states that we should pick the size of the interval and use the average of the sum of the images in the interval so that, after multiplication, we would reach a nice approximation. It then works with a series, and makes the number of trapezoids go to infinity. Basically, that would imply something like $\lim_{h->0} \sum_{\delta=0}^{\delta=b-a} (f(a+\delta) + f(a+\delta+h))(0.5 * h)$. If Niculescu [1] is correct, then there is an upper bound to the inequality just mentioned, which is a huge trapezoid made of the images of the endpoints of the interval. Because f is convex, all its images will, indeed, lie below the straight line joining $f(a)$ and $f(b)$ and, therefore, such will be true and we can majorize the calculation of the integral via $0.5(b-a)(f(a)+f(b))$. As for the midpoint rule, it is an approximation via midpoint image terms multiplied by twice the length of the slices taken, which, when to infinity, will finally lead to the integral value. It is not making much sense for what we have at the left side of the inequality, then. It is missing duplication of $(b-a)$, at least. However, easy to see that $(b-a)$ is less than twice it. With this, we would just need to justify why the average of the domain interval would be thought as minorant of the integral. We would then be looking for some function value we were sure to be less than the actual measures which should be used to calculate the integral via midpoint. It really is not looking reasonable to assume it would be the image of half the sum of the extremes of the interval of the domain. Therefore, if the upper bound may be easily explained like that, but then reminding us it is not a sharp bound, the lower bound cannot be justified this way with reasonable accuracy. It is not true the lowest image of every convex function is located in the middle of the interval of its domain. Isolated pieces of convex functions are still convex, one must remember.

In Dragomir and Pierce [3], page 7, we learn that Hermite did not pay much attention to the now so famous Hermite-Hadamard inequalities. He sent

a short note to a journal, and nobody has cared much about it, according to Dragomir and Pierce [3]. That would match what was seen so far about the inequalities: weak bounds with a lot of room for refinement.
It is also mentioned in Dragomir and Pierce [3] that Fejér would have generalized the result by Hermite, without even mention to him, to the following reading:

Theorem 2.2. *Consider the integral* $\int_a^b f(x)g(x)dx$*, where* f *is a convex function in the interval* (a,b) *and* g *is a positive function in the same interval such that*

$$g(a+t) = g(b-t), \quad 0 \leq t \leq 0.5(a+b),$$

i.e., $y = g(x)$ *is a symmetric curve with respect to the straight line which contains the point* $(0.5(a+b),0)$ *and is normal to the* $x-$*axis. Under those conditions:*

$$f\Big(0.5(a+b)\Big)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq (0.5f(a)+0.5f(b))\int_a^b g(x)dx.$$

For $g(x) \equiv 1$, $x \in (a,b)$, Dragomir and Pierce [3] remind us we will get the HH inequalities.
From Dragomir and Fitzpatrick [2], we copy the method below to determine the left bound for the inequality:

- Consider the midpoint situation in the definition of convexity:

$$f(0.5(x+y)) \leq 0.5f(x)+0.5f(y);$$

- (Problem 1) Now re-write x as $x := \lambda x + (1-\lambda)y$ and y as $y := (1-\lambda)x + \lambda y$;

- Integrate all over the variable λ, which ranges from 0 to 1 via definition, in the interval of definition;

- Then they would claim the result was achieved. We will follow their reasoning, but we are already making public we oppose to all of it. See:

$$f(0.5(x+y)) \leq 0.5f(\lambda x+(1-\lambda)y)+0.5f((1-\lambda)x+\lambda y)$$

$$\iff \int_0^1 f(0.5(x+y))d\lambda \leq 0.5\int_0^1 f(\lambda x+(1-\lambda)y)d\lambda +$$

$$+0.5\int_0^1 f((1-\lambda)x+\lambda y)d\lambda$$

(Problem 2)

$$\iff f(0.5(x+y)) \leq \frac{1}{y-x}\int_0^1 f(x)dx.$$

However, the problems with this proof are so extraordinary that it is unavoidable to think how it was thought to be plausible before. First of all, we had two free variables at the beginning, in the definition of convexity. Basically, they behave as belonging to different dimensions, that is, we hold a definition using the function twice, same function, what makes the inequality live at $\Re^2$, rather than $\Re$. The variables are made to live freely, therefore. It cannot be the case that, contrary to the intended definition, we simply make ties between them, so that there is exclusion of possibilities, without ever mentioning that or providing a way of reverting that process...it is all absurd. In (Problem 1), that is the issue. Basically, there would be a trial of making a single dimensional variable become an object in a 3-dimensional space (variables are: x, λ, y, first instance). That is perfectly insane, if not accompanied of a very good explanation and a way to revert the process. Just to provide the reader with simplest reasoning of all, with that assignment, once λ is chosen, and x is chosen, we are left with a single choice for y. However, before that, y would be freely chosen even after the first two choices...It is missing at least a correspondence function which allows us to progress and also revert the process to the initial variable at the end...the symbols used are applied in computation, Maple. However, even there, the process must be reversible...

In (Problem 2), the result is actually zero, for it is missing noticing it is $x-y$ in one of them, but it is $y-x$ in the other, and that obviously makes it all untrue, providing enough support to the thesis that the step noticed in (Problem 1) should never have been dared...

Reviewing other proofs is all very interesting...however, we must be able to then find a proof for the left bound if other proofs are mistaken. We are then left with some hope on Féjer's.

In not being able to access his paper online, we were left with our own intuition.

See: we are after a specific value, for the function f, such that its image will be less than or equal to all other values of the function, once one may keep in mind the approximation to integrals and the division for the interval as a good start. We are then after a minimum for the function, that is, of course, the lousiest approximation, just like the upper bound reasoning. The fact that any straight line between two points of the upwards interior of a convex curve cannot find any curve points above it makes us see that we cannot hold intermediary pieces of the curve which go up and then down once more. A convex curve cannot oscillate. That is true for each piece of it. A curve that is not symmetric, however, and goes down in an almost ninety degree angle to smooth at the end and go bowl shape, is definitely convex (another easy example is a twisted V sort of shape, with different angles. Notice that smoothness is not a requirement for integration, only continuity, and notice that HH inequality theorem actually misses the adequate analytical placement,

which we shall make later on here). Picking the middle point of this curve, however, would not mean lowest value for the function. For that reason, it is not really adequate to choose the middle point in the domain, unless we prove it is always, perhaps in average, less than the approximation via sums to the integral...
Of course, the point of the domain to which the derivative of the function is zero is our best candidate.
With the function specified, everything is easy. However, we wish to work with a generic sort of convex function.
In McAndrew and Dragomir [0], we find a better proof for the left bound for HH. There, it reads (call it PROOF Z):

Proof. Assume f is differentiable and convex in (a, b), then, for each $x, y \in (a, b)$, one has the inequality

$$f(x) - f(y) \geq (x - y)f'(y).$$

Using the property of modulus, we then get

$$f(x)-f(y)-(x-y)f'(y) = |f(x)-f(y)-(x-y)f'(y)| \geq ||f(x)-f(y)|-|x-y||f'(y)||,$$

for each $x, y \in (a, b)$. Now choose $y = 0.5(a + b)$ and replace y on previous inequality to get

$$f(x) - f(0.5(a + b)) - (x - 0.5(a + b))f'(0.5(a + b)) \geq$$
$$||f(x) - f(0.5(a + b))| - |x - 0.5(a + b)||f'(0.5(a + b))||,$$

for any $x \in (a, b)$. Now, integrating that all in $[a, b]$:

$$\int_a^b [f(x) - f(0.5(a + b)) - (x - 0.5(a + b))f'(0.5(a + b))]dx \geq$$
$$\int_a^b ||f(x) - f(0.5(a + b))| - |x - 0.5(a + b)||f'(0.5(a + b))||dx,$$

$$\int_a^b f(x)dx - \int_a^b f(0.5(a+b))dx - \int_a^b xf'(0.5(a+b))dx + 0.5(a+b)f'(0.5(a+b))(b-a) \geq$$
$$\int_a^b ||f(x) - f(0.5(a + b))| - |x - 0.5(a + b)||f'(0.5(a + b))||dx.$$

Dividing all by $(b - a)$, we then get:

$$\frac{1}{b-a}\int_a^b f(x)dx - f(0.5(a+b)) - \frac{1}{b-a}f'(0.5(a+b))\int_a^b xdx + 0.5(a+b)f'(0.5(a+b)) \geq$$
$$\frac{1}{b-a}\int_a^b ||f(x) - f(0.5(a + b))| - |x - 0.5(a + b)||f'(0.5(a + b))||dx.$$

or

$$\frac{1}{b-a}\int_a^b f(x)dx - f(0.5(a+b)) \geq$$

$$\frac{1}{b-a}\int_a^b ||f(x) - f(0.5(a+b))| - |x - 0.5(a+b)||f'(0.5(a+b))||dx \geq 0.$$

This closes our proof for good. □

In this paper, we have added ★ next to every new result we present, and ♠ for each old result we decided to mess up with a bit.

Remark 2. The above proof is excellent, but it makes us notice that any value, rather than $0.5(a+b)$, could have been used as y. What that gives us is that any value of the function may be used in the HH-inequality, without harm to result.

From the above proof, we infer:

★New extension for the HH inequality★

Theorem 2.3. *For any* $f : [a,b] -> \Re$*,* f *being convex and continuous in* $[a,b]$*, it is always true that*

$$f(\lambda a + (1-\lambda)b) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2},$$

for each pre-determined value of $\lambda \in [0,1]$*.*

Remark 3. The new theorem, above, will give us chance of using any value of D_f as lower bound, so that it is far more flexible and easy to be understood, sufficing finding out the maximum of the function in the interval to have the bound optimized.

Proof. Simply re-write the last proof replacing the value of y. It all sounds reasonable, once if it is valid for every y, it is also valid for a specific value of it. And doing this way, we solve (Problem 1), that of the dimensions fight. □

We also dare re-writing the own HH-inequality to include the essential condition, taken for granted, without spelling it out, in the theorem:

♠*Re-writing the HH-inequality* ♠

Theorem 2.4. *For $f : [a, b] - > \Re$, f being convex and continuous almost everywhere*[1]*, it is always true that*

$$\text{(In. 0)} \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}.$$

3. Terminology

We use the same symbols and definitions presented in Pinheiro [5]:

- K_s^1 for the class of S-convex functions in the first sense, some S;
- K_s^2 for the class of S-convex functions in the second sense, some S;
- K_0 for the class of convex functions;
- s_1 for the variable S, $0 < S \leq 1$, used in the first definition of S-convexity;
- s_2 for the variable S, $0 < S \leq 1$, used in the second definition of S-convexity.

Remark 4. The class of 1-convex functions is just a restriction of the class of convex functions, that is, when $X = \Re_+$,

$$K_1^1 \equiv K_1^2 \equiv K_0.$$

[1]Notice that the theorem is not valid otherwise. However, the definition of convex functions does not include the continuity condition, what means that isolated points forming a convex shape are also convex functions, for instance. Some books close the issue on continuity stating that the Riemann integral is only plausible for continuous functions and if there is any discontinuity one should go for the Lebesgue integral instead. However, it is common sense that if we can make our small Riemann intervals always pick the point where discontinuity occurs in the middle, for instance, it still works.

4. Definitions

Definition 3. *A function $f : X- > \Re$, $f \in C^1$, is said to be s_1-convex if the inequality*

$$f(\lambda x + (1-\lambda^s)^{\frac{1}{s}} y) \le \lambda^s f(x) + (1-\lambda^s) f(y)$$

holds $\forall \lambda \in [0,1]$, $\forall x, y \in X$ such that $X \subset \Re_+$.

Definition 4. *f is called s_2-convex, $s \neq 1$, if the graph lies below a 'bent chord' (L) between any two points, that is, for every compact interval $J \subset I$, with boundary ∂J, it is true that*

$$\sup_J (L - f) \ge \sup_{\partial J} (L - f).$$

Definition 5. *A function $f : X- > \Re$, in C^1, is said to be s_2-convex if the inequality*

$$f(\lambda x + (1-\lambda) y) \le \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds $\forall \lambda \in [0,1]$, $\forall x, y \in X$ such that $X \subset \Re_+$.

5. Explaining/criticizing/refereeing old results and producing new ones

In Dragomir and Fitzpatrick [2], we read precisely the following words:

Theorem 5.1. *Let f be a $s-$convex function[2] in the second sense on an interval $I \subset [0, \infty)$ and let $a, b \in I$ with $a < b$. Then:*

$$\text{(In. 1)} \quad 2^{s-1} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x)dx \le \frac{f(a)+f(b)}{s+1}.$$

Right side

Proof. First, we reproduce the proof for the right side:

- Applying the definition of s_2-convex function to the two extreme points of the interval of its definition, we have:

$$f((1-\lambda)a + \lambda b) \le (1-\lambda)^s f(a) + (\lambda^s) f(b), \quad \forall \lambda \in [0,1];$$

[2]Notice that we have added belonging to C^1 to our definition of s_2-convexity, so that we do not need to mention continuous almost everywhere in the theorem here.

- Now, as shown in Dragomir and Fitzpatrick [2], we integrate all in $[0,1]$, and on the variable λ, to achieve

$$\frac{1}{b-a}\int_0^1 f(x)dx \leq \frac{1}{s+1}f(a) + \frac{1}{s+1}f(b) = \frac{f(a)+f(b)}{s+1}.$$

□

Remark 5. The right hand side proof, presented above, seems perfect, in all senses, so that the result is good and sound. And it must also be best right bound possible because it is obtained straight from the definition.

Notice that if we apply the same reasoning to absolutely any value in the domain of the function, the result is the same, only swapping a and b with their adequate replacements, see:

$\star s_2$-convex functions and HH generalized$\star$

Theorem 5.2. *For any choice of couple of elements of the domain of any convex function, which is continuous almost everywhere, we have:*

$$\frac{1}{x_2-x_1}\int_{x_1}^{x_2} f(x)dx \leq \frac{f(x_1)+f(x_2)}{s+1}.$$

Right side

Proof. First, we reproduce the proof for the right side:

- (Step 1) Applying the definition of s_2-convex function to any two points of the interval of its definition, say x_1 and x_2, we have:

$$f((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda)^s f(x_1) + (\lambda^s) f(x_2), \quad \forall \lambda \in [0,1];$$

- Now, as shown in Dragomir and Fitzpatrick [2], we integrate all in $[0,1]$, and on the variable λ, to achieve

$$\frac{1}{x_2-x_1}\int_0^1 f(x)dx \leq \frac{1}{s+1}f(x_1) + \frac{1}{s+1}f(x_2) = \frac{f(x_1)+f(x_2)}{s+1}.$$

□

Still on the right side and another possible bound, however greater

The trapezoid rule for approximation of integrals actually states that we should pick the size of the interval and use the average of the sum of the images in the interval so that, after multiplication, we would reach a nice approximation. It then works with a series, and makes the number of trapezoids go to infinity. Basically, that would imply something like $\lim_{h->0}\sum_{\delta=0}^{\delta=b-a}(f(a+\delta)+f(a+\delta+h))(0.5*h)$. In the convex case, as dealt with earlier on in this paper, there is an upper bound, which is a huge trapezoid made of the images of the endpoints of the interval. Because f is convex, all its images will, indeed, lie below the straight line joining $f(a)$ and $f(b)$ and, therefore, such will be true and we can majorize the calculation of the integral via $0.5(b-a)(f(a)+f(b))$. If we were to do the same for s_2-convexity, however, we would have to make the straight line become a 'bent line', according to the usual behavior of the limiting curve for K_s^2. In this case, we would have $\lambda^s f(0.5(a+b))+(1-\lambda)^s f(0.5(a+b))$. The expression, when evaluated on the maximum size of it would then be $\frac{f(0.5(a+b))}{2^{s-1}}$ (check via derivative over λ, for instance, or in Pinheiro, [4], at *www.geocities.com/mrpprofessional*, submitted, but not yet published). Now, the big issue is...

Who wins the fight: geometric deduction or analytical?

Via Analysis:

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1}.$$

Via Geometry:

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(0.5(a+b))}{2^{s-1}}.$$

- Which one returns lowest upper bound/best refinement?
- Our question is then about the relationship: $\frac{f(a)+f(b)}{s+1}\ ?\ \frac{f(0.5(a+b))}{2^{s-1}}$.
- This question is not really easy to be answered if picked in an isolated manner, with no context, once not even the value for the image of the function being considered is the same. It all depends on the function itself, even in terms of derivatives. However, with a function as simple as $x^{0.5}$, and interval of domain $[0,1]$, we get the left bound is a better upper bound for the HH inequality. See:

$\frac{1}{1.5} \approx 0.6$? $\frac{0.5^{0.2} \approx 0.7}{2^{-0.5}}$
0.6 ? $0.7 * 2^{0.5} = 0.9$.

- Unfortunately, there is nothing more accurate than analytical deduction for boundaries, for eyes are less perfect than Mathematics...
  However, there might be cases in which the alternative is preferred, it is just a matter of doing research on them.

★Alternative to HH for s_2-convex functions, geometric★

Theorem 5.3. *Let f be an $S-$convex function*[3]*, in the second sense, in an interval $I \subset [0, \infty)$ and let $a, b \in I$ with $a < b$. Then:*

$$\text{(In. 1)} \quad 2^{s-1} f(0.5(a+b)) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq 2^{1-s} f(0.5(a+b)).$$

The inequality above looks gracious and nice to work with. However, one must remember that the left bound is not proven by us yet.
At this stage, one could wonder why nobody has applied (Step 1) to convexity yet. They probably have, we suppose, but just like the other results we hold, we simply think they have not yet done so because we probably did not do enough research on that...
However, let's try to do so, even if once more. Remember that a convex function is a special case of an s_2-convex function as well, where $s_2 = 1$.

New Hermite-Hadamard bound for convex functions

If we integrate on the definition of convexity, we get:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

$$\iff \frac{1}{x-y} \int_a^b f(x)dx \leq f(x) 0.5\lambda^2|_a^b + f(y)(b-a) - 0.5\lambda^2 f(y)|_a^b$$

$$\iff \frac{1}{x-y} \int_a^b f(x)dx \leq 0.5 f(x)(b^2-a^2) + f(y)(b-a) - 0.5 f(y)(b^2-a^2).$$

This way, it could not be easier to find a better right bound for the HH inequality. Comparing the bound above with the usual one, $(0.5(f(x)+f(y))$, we get:

$$0.5f(x)(b^2-a^2)+f(y)(b-a)-0.5f(y)(b^2-a^2) = 0.5(f(x)-f(y))(b^2-a^2)+f(y)(b-a)$$

[3]Notice that we have added belonging to C^1 to our definition of s_2-convexity, so that we do not need to mention continuous almost everywhere in the theorem here.

$$= [0.5(f(x) - f(y))(b + a) + f(y)](b - a)$$

Suppose the above is less than the usual bound:

$$[0.5(f(x) - f(y))(b + a) + f(y)](b - a) \leq 0.5(f(x) + f(y))$$

Now take $b = a + \lambda, \quad \lambda > 0$. And the expression above will be equal to

$$[0.5(f(x) - f(y))(2a + \lambda) + f(y)](\lambda) \leq 0.5(f(x) + f(y))$$

$$\iff [0.5(2af(x) - 2af(y) + \lambda f(x) - \lambda f(y)) + f(y)]\lambda \leq 0.5(f(x) + f(y))$$

$$\iff [af(x) - af(y) + 0.5\lambda f(x) - 0.5\lambda f(y) + f(y)]\lambda \leq 0.5(f(x) + f(y))$$

$$\iff (a + 0.5\lambda)\lambda f(x) + (1 - 0.5\lambda - a)\lambda f(y) \leq 0.5(f(x) + f(y))$$

So, all we need, for the above being true, is that: $(a + 0.5\lambda)\lambda \leq 0.5$ and $(1 - 0.5\lambda - a)\lambda \leq 0.5$, or:

$$\begin{cases} 0.5\lambda^2 + a\lambda - 0.5 \leq 0 \\ -0.5\lambda^2 - a\lambda + (\lambda - 0.5) \leq 0 \end{cases}$$

or $\lambda \leq 1$.
This basically means we have improved the upper bound of the HH inequality when the distance between a and b is at most one, what is fantastic for most of the analytical work available in the World now, all liking the interval $[0, 1]$ better than others.

$\star$New HH-inequality version with more refined upper bound for unit, or smaller, intervals$\star$

Theorem 5.4. *$f : [a, a + \lambda]- > \Re$, $\lambda \leq 1$, f being convex, it is always true that*

$$f\left(\frac{2a + \lambda}{2}\right) \leq \frac{1}{\lambda}\int_a^{a+\lambda} f(t)dt \leq (a + 0.5\lambda)\lambda f(x) + (1 - 0.5\lambda - a)\lambda f(y).$$

(In. 1), Left side
Following Dragomir and Fitzpatrick [2]:

$$f(0.5a + 0.5b) \leq 0.5^s f(a) + 0.5^s f(b).$$

Now, the trick in Dragomir and Fitzpatrick [2] consists of re-writing a and b as functions of λ, like this: $a := \lambda a + (1-\lambda)b$ and $b := (1-\lambda)a + \lambda b$. With this, we get our function in λ:

$$f(0.5(a+b)) \leq 0.5^s f(\lambda a + (1-\lambda)b) + 0.5^s f((1-\lambda)a + \lambda b).$$

From here, we do what we did before, integrating all over the variable λ in the interval $[0,1]$, in which it is defined for $S-$convexity in general.
We then get:

$$f(0.5(a+b)) \leq 0.5^s \frac{1}{a-b} \int_0^1 f(x)dx + 0.5^s \frac{1}{b-a} \int_0^1 f(x)dx = 0.$$

The result above is absurd. Dragomir and Fitzpatrick [2] bears an incredibly basic mistake, that of replacing the variable a, there called x with the sum of 3 variables, one being the own a. First of all, in order to do such, one would have to prove such is possible, that is, expressing precisely, with no excess or fault, the whole range of a, which is the range of the function, via the replacement expression. However, even after such is actually verified, trivially, one is stuck with a further problem, a and b are supposed to be free variables, running freely over the domain of the function, but with this replacement we create a tie between a and b, once it all depends on λ and, therefore, there will be exclusions of inequalities from the previously intended domain, what is not acceptable in the rigors of Mathematics. That is because once a value for λ is chosen to one of the right parcels, the other will be determined. However, it would be free originally to be whatever one intended as well, including the same value as a. Easy to see that a can never be in both parcels the way it would be re-stated, for instance. That is absurd choice and replacement, then. Therefore, if the left bound found in Dragomir and Fitzpatrick [2] is ever correct, it must be the case that we must find another explanation for that fact, another proof.
The other interesting remark which fits here is that there is actually no well-posedness application for the left bound, for a lower bound must be highest as possible. However, a convex function, for instance x^2, may be picked by any small interval, or large, and will still be convex there. That won't mean the middle point will always be highest, rather the opposite, most of the time. In the half which decreases, the highest point in the image is the first, therefore, $f(a)$. In the half which increases, the highest point is the last, therefore, $f(b)$. In the bottom slice, taken symmetrically, the highest point is either $f(a)$ or $f(b)$, the lowest being the middle point. Intuitively, it also sounds like Hermite-Hadamard inequality is not the best inequality possible for convex functions, actually their own assertion confirms it all: it is valid, but there might be better fit for the lower bound.

If we apply PROOF Z here as well, using the same steps, we will end up at the same conclusion, this time for f which is s_2-convex in (a,b), that is:

$$\frac{1}{b-a}\int_a^b f(x)dx - f(0.5(a+b)) \geq$$

$$\frac{1}{b-a}\int_a^b ||f(x)-f(0.5(a+b))| - |x-0.5(a+b)||f'(0.5(a+b))||dx \geq 0.$$

However, trivially, $2^{s-1} \leq 1$, once $0 < s \leq 1$. And, therefore:

$$\frac{1}{b-a}\int_a^b f(x)dx - 2^{s-1}f(0.5(a+b)) \geq \frac{1}{b-a}\int_a^b f(x)dx - f(0.5(a+b)) \geq 0.$$

So, we can prove, analytically, that the left bound is accurate. It is obviously also the case that this new bound, imposed on top of the convex one, is worse, for if s_2-convexity is more than proved to be a proper analytical real extension of convexity, the lower bound for convexity should be its best bound as possible.

So, the above is a true inference, as many others. However, if whatever is convex is also s_2-convex, it obviously does not make sense to change or think of changing the lower bound. Only the upper bound is worth pursuing then, when it comes to s_2-convexity, which is a geometric lift on the convexity geometric boundary, therefore only changing upper bounds, in a meaningful way, when compared to convexity. It is obvious deduction that everything which is less than a convex function area will also be less than an s_2 in the non-negative domain, once it was also proved that an s_2-convex function is non-negative.
This way, for K_s^2, we just repeat the same lower bound used for convexity, following the rigors of the well-posedness theory for mathematical/classical logic theorems.

$\star$Best HH for s_2-convex functions$\star$

Theorem 5.5. *Let f be an $S-$convex function*[4]*, in the second sense, in an interval $I \subset [0,\infty)$ and let $a,b \in I$ with $a < b$. Then:*

$$\textit{(In. 1')} \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1}.$$

[4]Notice that we have added belonging to C^1 to our definition of s_2-convexity, so that we do not need to mention continuous almost everywhere in the theorem here.

Generators for s_2-functions

In case a reader wishes to test the inequality against a few known models for $S-$convexity, we mention those already known, from Dragomir and Pierce [3], page 283:

For $0 < s < 1$, $\{a, b, c\} \in \Re$, $u \in \Re_+$, take M to be:

$$f(u) = a, \quad u = 0$$

and

$$f(u) = bu^s + c, \quad u > 0.$$

Further, if $b \geq 0$ and $c \leq a$ then $f \in K_s^1$, which we shall name A, that is, A will stand for set of conditions for which the functional model M is a generator of examples of functions in K_s^1.

Remark 6. (Problem Q) We will keep a line of Dragomir and Pierce [3] for future reference here so that we can refer to it,later on, for criticism and analysis. That line states that if A is found but $c < a$, that is, not allowing $c = a$, then we have a non-decreasing function in $(0, \infty]$ but not necessarily in $[0, \infty]$. This is a severely odd remark, once there is mathematical proof, also provided in Dragomir and Pierce [3], as to state that any function in K_s^1 will fall into the non-decreasing category for the $(0, \infty]$ case. We will then keep this by now and discuss this issue later on on this paper. In principle, this line should not be there at all.

The source Dragomir and Pierce [3], on page 292, also reveals that if $b \geq 0$ and $0 \leq c \leq a$ then $f \in K_s^2$. Call these conditions, for pertinence to K_s^2, B.

The same source, Dragomir and Pierce [3], on page 292, also reveals that if $b > 0$ and $c < 0$ then $f \notin K_s^2$.

Basically, then, if $b \geq 0$ and $0 \leq c \leq a$, we should have a function which is simply $S-$convex, for it belongs to both types of $S-$convexity.

An example of such a function then, good to use for tests for $S-$convexity in general, would be:

$$f(x) = x^{0.15}.$$

See: in this case, $f(0) = 0$, making $a = 0$. Also, $b = 1$ and $0 \leq c \leq a$ because $c = 0$ as well.

It is then understood that any exponent between 0 and 1 could be used in this situation and the result would always be an $S-$convex function.

From Pinheiro [6], we learn that any s_1-convex function is non-decreasing. With this, the easy left bound to be determined comes from:

$$\begin{aligned} f(x_1) \leq f(x), \quad & \forall x \in [x_1, b] \\ \Longleftrightarrow \int_{x_1}^{b} f(x_1)dx & \leq \int_{x_1}^{b} f(x)dx \\ \Longleftrightarrow f(x_1)(b - x_1) & \leq \int_{x_1}^{b} f(x)dx \\ \Longleftrightarrow f(x_1) & \leq \frac{1}{b - x_1}\int_{x_1}^{b} f(x)dx. \end{aligned}$$

Basically, we may choose any x_1 of our taste to start, even the most obvious one, which will generate less possibilities, but also lowest lower bound, when we should be looking for precisely the opposite, the highest one.
If we choose the easiest, we are left with:

$$f(a) \leq \frac{1}{b-a}\int_{a}^{b} f(x)dx.$$

In this paper, we shall stick to the easiest lower bound, for practical purposes, having explored the other sort of convexity and the usual one to exhaustion.
As for the upper bound, we may apply approximation via largest area. We will then make use of the formula $\lambda^s f(a) + (1 - \lambda^s)f(b)$. Once the function is non-decreasing, $f(a) \leq f(b)$. That allows us to majorize via $f(b)$, what is nice. The result easily comes:

$\star$HH-inequality for s_1-convex functions$\star$

Theorem 5.6. *Let f be a s_1-convex function*[5] *in an interval $I \subset [0, \infty)$ and let $a, b \in I$ with $a < b$. Then:*

$$\textit{(In. 2)} \quad f(a) \leq \frac{1}{b-a}\int_{a}^{b} f(x)dx \leq f(b).$$

[5]Notice that we have added belonging to C^1 to our definition of s_2-convexity, so that we do not need to mention continuous almost everywhere in the theorem here.

Remark 7. Notice that, because s_1-convexity is not a proper extension of convexity, only the non-decreasing convex functions could, possibly, fit inside of a formula used for s_1-convex functions (replacing $f \in K_s^1$ with f convex in the above HH-inequality version, we cannot, ever, use a decreasing function which is also convex, then).

6. Conclusion

In this paper, we left at least two questions unaddressed: best HH-type inequality for s_1-convexity and the issue about the conflicting information found in the literature regarding K_s^1 (Problem Q). On the top of these problems, arisen by this paper, there is still the definition of $S-$convexity inside of the complex case scenario, which we have not yet approached, but intend to, and the issue on the models for S-convex functions. The results provided in this paper, however, seem to have increased, very meaningfully, the existing scholarship in the area.

7. Bibliography

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