$H - H$ Inequality for s_1-convex Functions

I.M.R. Pinheiro

PO BOX 12396, A'Beckett st, Melbourne, Victoria, 8006, AUSTRALIA

e-mail: $mrpprofessional@yahoo.com$

Abstract: In this paper, we make a note on our previous results on HH and K_s^1. This note is correctional. We also extrapolate what was found in our paper with IJPAMS.
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1. Introduction

Our paper is divided into the following sections:

- Hermite-Hadamard Inequality itself;
- Terminology & definitions;
- Criticism on a few old findings regarding HH and s_1-convexity and a bit extra;
- Conclusion.

2. Hermite-Hadamard Inequality

In this paper, we have added ★ next to every new result we present, and ♠ for each old result. From [7], we copy:

♠Extended HH Inequality♠

Theorem 2.1. *For any $f : [a, b]- > \Re$, f being convex and continuous in $[a, b]$, it is always true that*

$$f(\lambda a + (1-\lambda)b) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2},$$

for each pre-determined value of $\lambda \in [0, 1]$.

Remark 1. The theorem above gives us the chance of using any value of D_f as lower bound, so that it is far more flexible and easier to be understood, sufficing finding out the maximum of the function in the interval to have the bound optimized.

3. Terminology & definitions

We use the same symbols and definitions presented in [5]:

- K_s^1 for the class of S-convex functions in the first sense, some S;
- K_s^2 for the class of S-convex functions in the second sense, some S;
- K_0 for the class of convex functions;
- s_1 for the variable S, $0 < S \leq 1$, used in the first definition of S-convexity;
- s_2 for the variable S, $0 < S \leq 1$, used in the second definition of S-convexity.

Remark 2. The class of 1-convex functions is just a restriction of the class of convex functions, that is, when $X = \Re_+$,

$$K_1^1 \equiv K_1^2 \equiv K_0.$$

Definition 3. *A function $f : X -> \Re$, $f \in C^1$, is said to be s_1-convex if the inequality*

$$f(\lambda x + (1 - \lambda^s)^{\frac{1}{s}} y) \leq \lambda^s f(x) + (1 - \lambda^s) f(y)$$

holds $\forall \lambda \in [0, 1]$, $\forall x, y \in X$ such that $X \subset \Re_+$.

Definition 4. *f is called s_2-convex, $s \neq 1$, if the graph lies below a 'bent chord' (L) between any two points, that is, for every compact interval $J \subset I$, with boundary ∂J, it is true that*

$$\sup_J (L - f) \geq \sup_{\partial J} (L - f).$$

Definition 5. *A function $f : X -> \Re$, in C^1, is said to be s_2-convex if the inequality*

$$f(\lambda x + (1 - \lambda) y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y)$$

holds $\forall \lambda \in [0, 1]$, $\forall x, y \in X$ such that $X \subset \Re_+$.

4. Criticizing old results and producing new ones

The result below is taken from [7]. Unfortunately, back then, we had not found out what was wrong with the proof of the major assertion by Dragomir et al. regarding K_s^1: K_s^1 is formed by non-decreasing functions. However, in revision before publication of the preprint of [6], we did find out how to justify our previous intuition on that result bearing mandatory mistake. We need then to re-write our previous results adequately in order to express our evolving through the concept study.
We now simply re-think and re-phrase our previous result, as well as its deduction, adding a few extras to it:
Variety 1:

- Consider that we hold an s_1-convex function, which is non-decreasing;
- The easy left bound to be determined then appears from:

$$\begin{aligned} & f(x_1) \leq f(x), \quad \forall x \in [x_1, b] \\ \iff & \int_{x_1}^{b} f(x_1)dx \leq \int_{x_1}^{b} f(x)dx \\ \iff & f(x_1)(b - x_1) \leq \int_{x_1}^{b} f(x)dx \\ \iff & f(x_1) \leq \frac{1}{b - x_1} \int_{x_1}^{b} f(x)dx; \end{aligned}$$

- Basically, we may choose any x_1 of our taste to start, even the most obvious one, which will generate less possibilities, but also lowest lower bound, when we should be looking for precisely the opposite, the highest one.
  If we choose the easiest, we are left with ($x \in [a, b]$):

$$f(a) \leq \frac{1}{b - a} \int_{a}^{b} f(x)dx.$$

Variety 2:

- Consider that we hold an s_1-convex function, which is non-increasing;
- The easy right bound to be determined then appears from:

$$\begin{aligned} & f(x_1) \geq f(x), \quad \forall x \in [x_1, b] \\ \iff & \int_{x_1}^{b} f(x_1)dx \geq \int_{x_1}^{b} f(x)dx \end{aligned}$$

$$\iff f(x_1)(b-x_1) \geq \int_{x_1}^{b} f(x)dx$$
$$\iff f(x_1) \geq \frac{1}{b-x_1}\int_{x_1}^{b} f(x)dx;$$

- Basically, we may choose any x_1 of our taste to start, even the most obvious one, which will generate less possibilities, but also highest upper bound, when we should be looking for precisely the opposite, the lowest one.
  If we choose the easiest, we are left with:
  $$f(a) \geq \frac{1}{b-a}\int_{a}^{b} f(x)dx.$$

In this paper, we shall stick to the easiest lower bound, for practical purposes, having explored the other sort of convexity, as well as the usual one, to exhaustion.
Variety 1:

- As for the upper bound, we may apply approximation via largest area;
- We will make use of the formula $\lambda^s f(a) + (1-\lambda^s)f(b)$ now. Once the function is non-decreasing, $f(a) \leq f(b)$;
- Majorize via $f(b)$. The result easily comes, as stated below.

Variety 2:

- As for the lower bound, we may apply approximation via smallest area;
- We will make use of the formula $\lambda^s f(a) + (1-\lambda^s)f(b)$ now. Once the function is non-increasing, $f(b) \leq f(a)$;
- Minorize via $f(b)$. The result easily comes, as stated below.

$\star$HH-inequality for s_1-convex monotonous functions$\star$

- **Variety 1**:

Theorem 5.1. *Let f be a non-decreasing s_1-convex function*[1] *in an interval $I \subset [0, \infty)$ and let $a, b \in I$ with $a < b$. Then:*

$$\text{(In. 1)} \quad f(a) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq f(b).$$

- **Variety 2**:

Theorem 5.2. *Let f be a non-decreasing s_1-convex function*[2] *in an interval $I \subset [0, \infty)$ and let $a, b \in I$ with $a < b$. Then:*

$$\text{(In. 2)} \quad f(b) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq f(a).$$

Remark 3. Notice that the above theorems also apply to Convexity, which is a special case of s_1-convexity, suffices writing $s = 1$.

5. Conclusion

In this paper, we have produced an errata for our previous version of HH-inequality for the first sense of S-convexity. The amendment was made necessary due to revision of our review and further findings regarding invalidation of old statements about it, statements present in the literature for very long, unfortunately. We ended up with different versions of the HH-inequality for each case of monotonicity in K_s^1.

[1]Notice that we have added belonging to C^1 to our definition of s_1-convexity, so that we do not need to mention continuous almost everywhere in the theorem here.

[2]Notice that we have added belonging to C^1 to our definition of s_1-convexity, so that we do not need to mention continuous almost everywhere in the theorem here.

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