Lazhar's inequalities and the S-convex phenomenon

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Abstract: In this further little article, we simply extend Lazhar's work on inequalities for convex functions to those a little bit beyond: S-convex functions.
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1. Introduction

We seem to have developed the precursor, and so honorable, work of Professors Hudzik and Maligranda to a palatable level of suitability, for applications in diverse areas, by making their theory more foundational in the pure scope of the Science. In this further work, we wish to extend Lazhar's work to S-convexity functions.

V. I. Professor Lazhar has made use, as seen on [3], of the sources [1], [2], [6]. We obviously simply trusted Professor Lazhar's citations, refereed by the editorial board of JIPAM.

Little by little, the use of S-convexity is proven. By our extensions of results and foundational works, we have developed many tools that may be used in Optimization when dealing with functions that almost look like convex functions but are not. By splitting the domain of the function into intervals, one may make the whole function passive of work in Optimization with little effort.

In the next section, the set of symbols here used is explained in detail. **Section 3** will bring the results exposed by Lazhar in his precursor work. **Section 4** brings our new theorems, results derived from the extension of Lazhar's theorems to S-convexity, along with their proofs.

We use the symbols defined in [5]:

- K_s^1 for the class of S-convex functions in the first sense, some S;
- K_s^2 for the class of S-convex functions in the second sense, some S;
- K_0 for the class of convex functions;
- s_1 for the constant S, $0 < S < 1$, used in the first definition of S-convexity;
- s_2 for the constant S, $0 < S < 1$, used in the second definition of S-convexity.

3 Definitions

We use the definitions presented in [5]:

Definition 1. A function $f : X- > \Re$, f continuous (see [1] for argumentation), is said to be s_1-convex if the inequality

$$f(\lambda x + (1 - \lambda^s)^{\frac{1}{s}} y) \leq \lambda^s f(x) + (1 - \lambda^s) f(y)$$

holds $\forall\lambda \in [0,1]$, $\forall x, y \in X$ such that $X \subset \Re_+$.

Definition 2. f is called s_2-convex, $s \neq 1$, if the graph lies below a 'bent chord' (L) between any two points, that is, for every compact interval $J \subset I$, with boundary ∂J, it is true that $\sup_J(L-f) \geq \sup_{\partial J}(L-f)$.

Definition 3. A function $f : X- > \Re \in C^1$ is said to be s_2-convex if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds $\forall\lambda \in [0,1]$, $\forall x, y \in X$ such that $X \subset \Re_+$.

4 Lazhar's precursor theorems

Theorem 4.1. *If f is a convex function and $x_1, x_2, ..., x_n$ lie in its domain, $n \in N$, $n > 1$, then*[1]*:*

$$\sum_{i=1}^{n} f(x_i) - f\left(\frac{x_1 + x_2 + ... + x_n}{n}\right)$$

$$\geq \frac{n-1}{n}\left[f\left(\frac{x_1+x_2}{2}\right) + ... + f\left(\frac{x_{n-1}+x_n}{2}\right) + f\left(\frac{x_n+x_1}{2}\right)\right].$$

Theorem 4.2. *If f is a convex function and $a_1, ..., a_n$ lie in its domain, $n \in N$, $n > 1$, then*[2]*:*

$$(n-1)[f(b_2) + ... + f(b_n)] \leq n[f(a_1) + ... + f(a_n) - f(a)],$$

[1] We have added the information, which we believe to be essential, based on well-posedness theory for Philosophy, to the theorem. If the index is not natural and starts from 2, we do get problems.

[2] We have added the information, which we believe to be essential, based on well-posedness theory for Philosophy, to the theorem. If the index is not natural and starts from 2, we do get problems.

where $a = \frac{a_1+...+a_n}{n}$ *and* $b_i = \frac{na-a_i}{n-1}$, $i = 1, ..., n$.

5 Our theorems: extensions of Lazhar's work to S-convex functions

As a conclusion, for this one more precursor paper, we mention our own results, all based on Lazhar's previous developments.

Theorem 5.1. *If* f *is an* S_1*-convex function, which is also non-negative, and* $x_1, x_2, ..., x_n$ *lie in its domain, then*

$$\sum_{i=1}^{n} f(x_i) - f\left(\frac{x_1+...+x_n}{n^{\frac{1}{s}}}\right)$$

$$\geq \frac{n-1}{n}\left[f\left(\frac{x_1+x_2}{2^{\frac{1}{s}}}\right) + ... + f\left(\frac{x_{n-1}+x_n}{2^{\frac{1}{s}}}\right) + f\left(\frac{x_n+x_1}{2^{\frac{1}{s}}}\right)\right].$$

Proof. Using the condition of S_1-convexity, with $t = \frac{1}{2^{\frac{1}{s}}}$, we obtain:

$$f\left(\frac{x_1+x_2}{2^{\frac{1}{s}}}\right) + ... + f\left(\frac{x_{n-1}+x_n}{2^{\frac{1}{s}}}\right) + f\left(\frac{x_n+x_1}{2^{\frac{1}{s}}}\right)$$

$$\leq f(x_1) + f(x_2) + ... + f(x_n).$$

However,

$$\sum_{i=1}^{n} f(x_i) = \frac{n}{n-1}\sum_{i=1}^{n} f(x_i) - \frac{1}{n-1}\sum_{i=1}^{n} f(x_i),$$

$$\sum_{i=1}^{n} f(x_i) = \frac{n}{n-1}\left[\sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n}\frac{1}{n}f(x_i)\right].$$

Replacing $\sum_{i=1}^{n} f(x_i)$ with its equivalent expression, as above, one gets:

$$f\left(\frac{x_1+x_2}{2^{\frac{1}{s}}}\right)+...+f\left(\frac{x_{n-1}+x_n}{2^{\frac{1}{s}}}\right)+f\left(\frac{x_n+x_1}{2^{\frac{1}{s}}}\right)$$

$$\leq \frac{n}{n-1}\left[\sum_{i=1}^{n} f(x_i)-\sum_{i=1}^{n}\frac{1}{n}f(x_i)\right].$$

With the subsequent application of the condition of S_1-convexity, one gets:

$$f\left(\frac{x_1+x_2}{2^{\frac{1}{s}}}\right)+...+f\left(\frac{x_{n-1}+x_n}{2^{\frac{1}{s}}}\right)+f\left(\frac{x_n+x_1}{2^{\frac{1}{s}}}\right)$$

$$\leq \frac{n}{n-1}\left[\sum_{i=1}^{n} f(x_i)-f(\frac{1}{n^{\frac{1}{s}}}\sum_{i=1}^{n}x_i)\right].$$

□

Theorem 5.2. *If f is an S_2-convex function, which is also non-negative, and $x_1, x_2, ..., x_n$ lie in its domain, then:*

$$\sum_{i=1}^{n} f(x_i)-f\left(\frac{x_1+...+x_n}{n}\right)$$

$$\geq \frac{2^{s-1}(n^s-1)}{n^s}\left[f\left(\frac{x_1+x_2}{2}\right)+...+f\left(\frac{x_{n-1}+x_n}{2}\right)+f\left(\frac{x_n+x_1}{2}\right)\right].$$

Proof. Using the condition of S_2-convexity, with $t=\frac{1}{2}$, we obtain:

$$f\left(\frac{x_1+x_2}{2}\right)+...+f\left(\frac{x_{n-1}+x_n}{2}\right)+f\left(\frac{x_n+x_1}{2}\right)$$

$$\leq 2^{1-s}(f(x_1)+f(x_2)+...+f(x_n)).$$

However,

$$\sum_{i=1}^{n} f(x_i) = \frac{n^s}{n^s-1}\sum_{i=1}^{n} f(x_i) - \frac{1}{n^s-1}\sum_{i=1}^{n} f(x_i),$$

$$\sum_{i=1}^{n} f(x_i) = \frac{n^s}{n^s-1}\left[\sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n} \frac{1}{n^s} f(x_i)\right].$$

Replacing $\sum_{i=1}^{n} f(x_i)$ with its equivalent expression, as above, one gets:

$$f\left(\frac{x_1+x_2}{2}\right) + ... + f\left(\frac{x_{n-1}+x_n}{2}\right) + f\left(\frac{x_n+x_1}{2}\right)$$

$$\leq 2^{1-s}\frac{n^s}{n^s-1}\left[\sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n} \frac{1}{n^s} f(x_i)\right].$$

With the subsequent application of the condition of S_2-convexity, one gets:

$$f\left(\frac{x_1+x_2}{2}\right) + ... + f\left(\frac{x_{n-1}+x_n}{2}\right) + f\left(\frac{x_n+x_1}{2}\right)$$

$$\leq 2^{1-s}\frac{n^s}{n^s-1}\left[\sum_{i=1}^{n} f(x_i) - f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)\right].$$

□

Remark 1. Considering the extended theorem for K_s^1 and $n = 3$, we get:

$$f(x_1) + f(x_2) + f(x_3) - f\left(\frac{x_1+x_2+x_3}{3^{\frac{1}{s}}}\right)$$

$$\geq \frac{2}{3}\left[f\left(\frac{x_1+x_2}{2^{\frac{1}{s}}}\right) + f\left(\frac{x_2+x_3}{2^{\frac{1}{s}}}\right) + f\left(\frac{x_3+x_1}{2^{\frac{1}{s}}}\right)\right].$$

Remark 2. Considering the extended theorem for K_s^2 for $n = 3$, we get:

$$f(x_1) + f(x_2) + f(x_3) - f\left(\frac{x_1+x_2+x_3}{3}\right)$$

$$\geq 2^{s-1}\frac{3^s-1}{3^s}\left[f\left(\frac{x_1+x_2}{2}\right)+f\left(\frac{x_2+x_3}{2}\right)+f\left(\frac{x_3+x_1}{2}\right)\right].$$

Theorem 5.3. *If f is an S_1-convex function, also non-negative, and $a_1, ..., a_n$ lie in its domain, then:*

$$(n^s-1)[f(b_1)+...+f(b_n)] \leq n^s[f(a_1)+...+f(a_n)]-nf(a),$$

$$\textit{where } a=\tfrac{a_1+...+a_n}{n^{\frac{1}{s}}} \textit{ and } b_i=\tfrac{n^{\frac{1}{s}}a-a_i}{(n-1)^{\frac{1}{s}}},\ i=1,...,n.$$

Proof. We now use the extended Jensen inequality ([4]):

$$f(b_1)+...+f(b_n) \leq f(a_1)+...+f(a_n),$$

and so,

$$f(b_1)+...+f(b_n)$$

$$\leq \frac{n^s}{n^s-1}[f(a_1)+...+f(a_n)]-\frac{1}{n^s-1}[f(a_1)+...+f(a_n)],$$

or

$$f(b_1)+...+f(b_n)$$

$$\leq \frac{n^s}{n^s-1}[f(a_1)+...+f(a_n)]-\frac{n}{n^s-1}\left[\frac{1}{n}f(a_1)+...+\frac{1}{n}f(a_n)\right].$$

Applying Jensen extended inequality, we get:

$$f(b_1)+...+f(b_n)$$

$$\leq \frac{n^s}{n^s-1}[f(a_1)+...+f(a_n)] - \frac{n}{n^s-1}\left[f\left(\frac{a_1+...+a_n}{n^{\frac{1}{s}}}\right)\right].$$

□

Theorem 5.4. *If f is an S_2-convex function, also non-negative, and $a_1,...,a_n$ lie in its domain, then*

$$(n-1)^s[f(b_1)+...+f(b_n)] \leq n[f(a_1)+...+f(a_n)] - n^s f(a),$$

where $a = \frac{a_1+...+a_n}{n}$ *and* $b_i = \frac{na-a_i}{(n-1)}$, $i = 1,...,n$.

Proof. We now use the extended Jensen inequality ([4]):

$$f(b_1)+...+f(b_n) \leq \frac{n-1}{(n-1)^s}(f(a_1)+...+f(a_n)),$$

and so,

$$f(b_1)+...+f(b_n)$$

$$\leq \frac{n}{(n-1)^s}[f(a_1+...+f(a_n)] - \frac{1}{(n-1)^s}[f(a_1)+...+f(a_n)],$$

or

$$f(b_1)+...+f(b_n)$$

$$\leq \frac{n}{(n-1)^s}[f(a_1)+...+f(a_n)] - \frac{n^s}{(n-1)^s}\left[\frac{1}{n^s}f(a_1)+...+\frac{1}{n^s}f(a_n)\right],$$

Applying Jensen extended inequality, we get:

$$f(b_1)+...+f(b_n)$$

$$\leq \frac{n}{(n-1)^s}[f(a_1)+...+f(a_n)] - \frac{n^s}{(n-1)^s}\left[f\left(\frac{a_1+...+a_n}{n}\right)\right],$$

$\square$

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