REVIEW ARTICLE ON s_1-CONVEXITY

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Long title: Getting the most basic facts on s_1-convexity correctly.

Abstract. We referee a few of the results stated in [2, PEARCE], mentioned as originated in [1, HUDZIK] in this paper, nullify others.

1. Introduction

In [2, PEARCE], page 283, we read that s_1-convex functions are non-decreasing. We also find a remark elucidating that this might be true for $(0, \infty)$, but not necessarily for $[0, \infty]$. This is at least an extremely odd remark. The proof there has got a step which is not so clear either. In this paper, we go through the proof presented in [2, PEARCE] in detail, on the hope we finally get to study this special sort of function, apparently wisely chosen by Hudzik and Maligranda (see [1, HUDZIK], for instance) to supplement s_2-convexity. Whilst the s_2-class, K_s^2, seems to bear perfect analytical objects, it seems not to be enough powerful to deal with certain basic mathematical ideas for which convexity is suitable. Notice that some concepts in Mathematics will deal with the limit line and that the sum 'one', in the image of the function, makes them easier, whilst others will not deal with the limit line, and the sum 'one', in the domain of the function, will make those easier.
This paper major intents are studying the rate of growth implications attached to the s_1-convex classification in the most primary level as possible.
This paper is organized as follows: **Section 1** brings the introduction, **Section 2** brings notations and definitions, **Section 3** analyzes the issues regarding K_s^1, **Section 4** brings our conclusions, and the last section, **Section 5**, our references.

2. Notations and Definitions

2.1. **Notations.** We use the symbology defined in [4, PINHEIRO]:

- K_s^1 for the class of $S-$convex functions in the first sense, some s;
- K_s^2 for the class of $S-$convex functions in the second sense, some s;
- K_0 for the class of convex functions;
- s_1 for the variable S, $0 < s_1 \leq 1$, used for the first type of S-convexity;
- s_2 for the variable S, $0 < s_2 \leq 1$, used for the second type of s-convexity.

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Remark 1. The class of 1-convex functions is simply a restriction of the class of convex functions, which is attained when $X = \Re_+$,

$$K_1^1 \equiv K_1^2 \equiv K_0.$$

2.2. **Definitions.** We use the definitions presented in [4, PINHEIRO] in what regards $S-$convexity, as well as in [3, PINHEIRO], in what regards convexity:

Definition 1. *$f : I- > \Re$ is considered convex iff*

$$f[\lambda x + (1-\lambda)y] \leq \lambda f(x) + (1-\lambda)f(y)$$

$\forall x, y \in I, \lambda \in [0,1]$.

Definition 2. *A function $f : X- > \Re$ is said to be s_1-convex if the inequality*

$$f(\lambda x + (1-\lambda^s)^{\frac{1}{s}} y) \leq \lambda^s f(x) + (1-\lambda^s)f(y)$$

holds $\forall \lambda \in [0,1]$; $\forall x, y \in X$; $X \subset \Re_+$.

Remark 2. If the complementary concept is verified, then f is said to be s_1-concave.

Definition 3. *A function $f : X- > \Re$ is called s_2-convex, $s \neq 1$, if the graph lies below a 'bent chord' (L) between any two points, that is, for every compact interval $J \subset I$, with boundary ∂J, it is true that*

$$sup_J(L-f) \geq sup_{\partial J}(L-f).$$

Definition 4. *A function $f : X- > \Re$ is said to be s_2-convex if the inequality*

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds $\forall \lambda \in [0,1]$; $\forall x, y \in X$; $X \subset \Re_+$.

Remark 3. If the complementary concept is verified, then f is said to be s_2-concave.

3. Remarks on the proof that K_s^1 is made of non-decreasing functions

CLAIM 1 (see [2, PEARCE, p.283]): **An s_1-convex function is always non-decreasing in $(0,\infty)$, but not necessarily in $[0,\infty)$.**

We copy the proof from [2, PEARCE] in order to make suitable remarks about it.
The proof, for the result, found there is:
"We have, for $u > 0$ and $\alpha \in [0,1]$:

$$\text{(Eq 1)} \qquad f[(\alpha^{\frac{1}{s}} + (1-\alpha)^{\frac{1}{s}})u] \leq \alpha f(u) + (1-\alpha)f(u) = f(u)\text{''}.$$

We here observe that Hudzik et al.'s preference, for the notation of the coefficients, was made to facilitate the result at the right end side of the inequality, what does make sense. Also, what they think of is that if a definition is suitable for any two members of the domain, it has to be suitable for the same member twice used. This is a minimum requirement, so that if it does not work for this case, it won't work for any others. Whatever conditions are found here must

be minimum conditions for all other cases.
Criticisms: There is obviously no sense in using the same member of the domain twice in one essentially geometric definition, always referring to limit curves, like in convexity. However, one could use that reasoning as a draft for work with limits. Such refinement, however, will keep their results, in terms of the K_s^1 group being formed by non-decreasing functions. Suffices, then, considering the second point as close to the first as wanted, so that there is no essential difference, given continuity, between the second and the first point considered.
We actually kill any chances of such claim by Dragomir et al. being true with the following Lemma:

Lemma 1. *In the definition of any sort of $S-$convexity, it is found, as basic enthymeme, that one cannot, ever, possibly, hold $x = y$, for the soundness of their theory.*

Proof. Easy examples of s_1-convex functions, which is also decreasing, are found (Take, for instance, $f(x) = -\frac{1}{10000}x^2 + \frac{1}{100}x$ in $y \geq \frac{2ax(1-a^s)^{\frac{1}{s}}}{(1-a^s)-(1-a^s)^{\frac{2}{s}}}$). Therefore, such a statement cannot, ever, be proven true, for validity of own Mathematics. We now hold an essential problem with a proof we, ourselves, got confused about, and even claimed to have refereed, at some stage, proof of Dragomir et al., which we actually made equivocated use of at $[6, PINHEIRO]$. Even though the counter-example proves the fact, we do need to find a fallacy with the proof, which is analytical. Basically, we work there with approximations, fact disregarded by Dragomir et al. in their report of the proof. In being S-convexity a majorly geometric definition, as much as convexity is, it is fundamental to hold at least two points in the reals, and such has to be accompanied by a multitude of them, therefore making it impossible to hold $x = y$. The proof is also of doubtful nature if made to check on consistency of the convexity definition, for instance. However, we explain the fact via continuity and irrelevance of figures coming after the decimal mark, making the values 'the same'.
We hold several options to go about the proof of the above Lemma:

- Point of proof 1: Even for convexity, making $x = y$ in the definition statement, seems to be analytically unsound. Notice that we hold two coefficients for the domain points. One 'takes' what the other 'puts in', basically, making use of 1 size as basis. The major question to be asked then is whether we could take so little from each extreme, at a point of making them be the same, once no mathematical formulae would be well-posed if using x, y to mean only x. There is obviously assumption of 'necessity' or 'imperative of force' there. This is the explanation, or justification, via well-posedness theory;
- Point of proof 2: There is 1, which appears there as figure to measure distance between points of the domain picked by formulae as basis. The fact obvious knocks down any trial of making x converge to y, or vice-versa. Basically, not even in Convexity, should one make use of such a reasoning. It all sounds equivocated. That is the distance between domain points being used as argumentation;

- Point of proof 3: There seems to be 'algebraic' allowance for a person to assume $x = y$ in the domain point, which is supposed to actually mean point between two other points (x and y in the formulae), that is, it seems 'algebraically' sound to do so. When writing $x = y$ in the formulae, we actually notice that, for any value of a picked, only one of the variables will remain inside of brackets, validating that reasoning. However, analytically, one cannot think of such. The analytical definition is matched to a geometric definition, which is clear as to the necessity of an 'interval', which is non-degenerated, in which to measure a function as to its pertinence to the S-convexity group. No inconsistencies can be allowed in Analysis. Dragomir et al. seems to change distance 1 into distance 0 between two supposedly different points of the domain of the function. However, 1 has to do with same line of reasoning as that of proportion, or 'scaling'. Can one propose a 0 unit factor for scaling? Do not think so...;
- Point of proof 4: The consistency of Mathematics guarantees that x must be fully different from y. S-convexity is supposed to be an extension of convexity, not mattering its sense. Any extension must guarantee inclusion of whatever is being extended...they both include Convexity algebraically. Allowing $x = y$ in their formulae, however, makes of every convex function a non-decreasing function with $f(0) \geq 0$, what is absurd. Another obvious thing is that, for s_1, not even algebraically possible such is, for there is no possible value for a in that situation.

Following any of the lines of reasoning above will easily justify the assertion contained in our Lemma. □

Interesting enough and worth making a remark is:

Remark 4. With K_s^1 sort of functions, one does not locate points directly on the real line, in what regards the domain, like it is easily done via convexity relation or even functions in K_s^2: One needs to create a correspondence in order to find out where, in the straight real line, one is whilst using the expression for domain points location in K_s^1. See: First we pick α for convexity, that is, determining precisely a geometric location on the real line between x and y. This way: $ha(\alpha) = \alpha x + (1-\alpha)y$. To this α, it will correspond a λ^s in s_1 convexity, and, therefore, our real position λ, which is severely delayed in relation to α, meaning proximity to the domain member accompanying the term λ.

The next step taken there is analyzing the function created to define the domain members to be used. This is then called $h(\alpha)$.

$$h(\alpha) = \alpha^{\frac{1}{s}} + (1-\alpha)^{\frac{1}{s}}.$$

Hudzik et al. then makes several claims about the function h above.
CLAIM 2 $h(\alpha)$ is a continuous function in $[0,1]$.
Remarks: Such is true because in the piece of the real domain mentioned there is no interruption of image assignment for the function.

CLAIM 3 The function $h(\alpha)$ is decreasing in $[0, 0.5]$ and increasing in $[0.5, 1]$.

$$h'(\alpha) = \frac{1}{s}\alpha^{\frac{1}{s}-1} - \frac{1}{s}(1-\alpha)^{\frac{1}{s}-1}.$$

Remarks: Notice that up to $\alpha = 0.5$, negativity goes better, and $h' \leq 0$. However, after $\alpha = 0.5$ then positivity does better, making $h' \geq 0$. That determines the function will grow after $\alpha = 0.5$ and will decrease before that mark. This simply confirms the early results of [1, HUDZIK], relying on what is mentioned at [2, PEARCE].

OUR CLAIM There is a minimum for $h(\alpha)$ in $\alpha = 0.5$, that is, in $h = 2^{1-\frac{1}{s}}$.

Proof: There, the function goes from decreasing to increasing. Therefore, we may say that to $\alpha = 2^{1-\frac{1}{s}}$, it corresponds a minimum result for the function attached to the value of s. Notice that when $s = 1$, $h = 1$, and when $s = 0$, $h = 0$.

OUR CLAIM Notice that what is being analyzed here is the domain bits we consider, not the image of the function. It does not really make much analytical sense worrying about how we walk over the domain points. If anything, only the image should be a concern, or the limit curves.

On top of that, what follows (Eq. 2), in [2, PEARCE], seems to be an excess which does not make much sense, besides the range remark, which also is not so clear.

4. Conclusion

In this short research note, we have reviewed the works of Hudzik et al. as portrayed in [2, PEARCE].

We actually reached the conclusion that K_s^1 cannot, possibly, ever, be formed by non-decreasing functions exclusively. On top, there is no sense in excluding only zero from the domain of non-decreasing images, for obvious analytical reasons, if continuity is ever implied.

If the function is defined in the whole interval, from zero to infinity (or to the Sup Infta, as we point out at $[5, PINHEIRO]$), and it is continuous, it cannot, trivially, be the case that it presents different behavior only at the extreme of the interval...the behavior is simply extended to the extreme for assumption of continuity, so that the remark cannot make any sense unless we allow a function of this group to be discontinuous. However, it is necessary to make mention that we did allow that to happen in order for the remark to make any sense.

5. References

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