S-convexity of balls and fixed-point theorems for mappings with nonexpansive square - ethical plagiarism and extension

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Abstract: – The purpose of this paper is to extend the previous results by Dr. K. Goebel, published by Compositio Mathematica, to S-convexity.

Key-words [1]*:* convex, S-convex, space, set, sequence

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1 Introduction

Our introduction gets developed via four major logical trends:

1. Basic definitions;
2. Basic symbology;
3. Sources used in this text;
4. Generalized index of sections for this paper.

In what follows, we simply try to organize the paper presentation with clarity.

1. Basic definitions:
   The concept of S-convexity is split into two notions, which are described below [5]:

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Definition 1. A function $f : X- > \Re$ is said to be S-convex in the first sense if

$$f(\lambda x + (1-\lambda^s)^{\frac{1}{s}} y) \leq \lambda^s f(x) + (1-\lambda^s) f(y),$$

$\forall x, y \in X$ and $\forall \lambda \in [0,1]$, where $X \subset \Re_+$.

Definition 2. A function $f : X- > \Re$ is said to be S-convex in the second sense if

$$f(\lambda x + (1-\lambda) y) \leq \lambda^s f(x) + (1-\lambda)^s f(y),$$

$\forall x, y \in X$ and $\forall \lambda \in [0,1]$, where $X \subset \Re_+$.

2. Basic symbology:

   - In this paper, we mean that f is an S-convex function in the first sense by saying that $f \in K_s^1$;
   - We use the same reasoning for a function g, S-convex in the second sense, and say then that $g \in K_s^2$;
   - We name s_1 the generic class constant for those functions that are S-convex in the first sense;
   - We name s_2 the generic class constant for those functions that are S-convex in the second sense;
   - B represents an arbitrary Banach space;
   - $|| \; ||$ represents the norm for B;
   - θ is the zero element in B;
   - x, y, z are elements in B;
   - $d(X)$ represents the diameter of a set $X \subset B$;
   - K_r the ball with radius r and centered in θ.

3. Sources used in this text as basis, including for symbols above, are basically [5] and [1].

4. The index for this paper includes:

   - Introduction;
   - Results regarding convexity;
   - Extension of definitions from convexity to $S-$convexity and some basic results as well;
   - Extension of main results;
   - Application;
   - Conclusion.

2 Results regarding convexity

The results below are mentioned in [1] as having originated from [2], [0], [3], and [2], respectively. The text below is not a faithful copy of the English words used there, however (only the mathematical symbols are cut and pasted).

Definition 3. B is called uniformly convex iff for every positive number ϵ, there exists a positive number δ such that for arbitrary $x, y \in K_1$, the inequality

$$||x - y|| \geq \epsilon$$

implies

$$||\frac{x + y}{2}|| \leq 1 - \delta.$$

Definition 4. B has normal structure iff every bounded and convex subset X, of B, containing more than one point, contains a nondiametral point, i.e., a point such that $\sup[||x - y|| : y \in X] < d(X)$.

Definition 5. The space is called uniformly non-square iff there is a positive number δ such that there are no $x, y \in K_1$ such that

$$||\frac{x - y}{2}|| > 1 - \delta$$

and

$$||\frac{x + y}{2}|| > 1 - \delta,$$

at the same time.

Definition 6. B is called strictly convex iff there are no segments laying on the boundary of K_1.

2.1 Modulus and Characteristic of Convexity

The definition and the results found below are mentioned in [1].

Definition 7. The modulus of convexity of B is the function $\delta : [0, 2]- > [0, 1]$ defined by the following formula
$\delta(\epsilon) = \inf \left[1 - ||\frac{x+y}{2}|| : x, y \in K_1,\ ||x - y|| \geq \epsilon \right].$

Theorem 2.1. *The function $\delta(\epsilon)$ is nondecreasing.*

Definition 8. The characteristic of convexity of B is the number $\epsilon_0 = \sup[\epsilon : \delta(\epsilon) = 0]$.

3 Extension of definitions to $S-$convexity

3.1 To s_1-convexity:

Definition 9. B is called uniformly s_1-convex iff for every positive number ϵ, there exists a positive number δ such that for arbitrary $x, y \in K_1$, the inequality

$$||x - y|| \geq \epsilon$$

implies

$$||\frac{x + y}{2}|| \leq 1 - \delta.$$

Definition 10. B has normal structure iff every bounded and s_1-convex subset X, of B, containing more than one point, contains a nondiametral point, i.e., a point such that $\sup[||x - y|| : y \in X] < d(X)$.

Definition 11. The space is called uniformly non-square iff there is a positive number δ such that there are no $x, y \in K_1$ such that

$$||\frac{x - y}{2}|| > 1 - \delta$$

and

$$||\frac{x + y}{2}|| > 1 - \delta,$$

at the same time.

Definition 12. B is called strictly s_1-convex iff there are no curves of the sort $\lambda^s p_1 + (1 - \lambda^s) p_2$ laying on the boundary of K_1, where p_1 and p_2 are both points in B.

3.2 To s_2-convexity:

Definition 13. B is called uniformly s_2-convex iff for every positive number ϵ, there exists a positive number δ such that for arbitrary $x, y \in K_2$, the inequality

$$||x - y|| \geq \epsilon$$

implies

$$||\frac{x + y}{2^s}|| < 2 - \delta.$$

Definition 14. B has normal structure iff every bounded and s_2-convex subset X, of B, containing more than one point, contains a nondiametral point, i.e., a point such that $\sup[||x - y|| : y \in X] < d(X)$.

Definition 15. The space is called uniformly non-square iff there is a positive number δ such that there are no $x, y \in K_1$ such that

$$||\frac{x-y}{2}|| > 1-\delta$$

and

$$||\frac{x+y}{2}|| > 1-\delta,$$

at the same time.

Definition 16. B is called strictly s_2-convex iff there are no curves of the sort $\lambda^s p_1 + (1-\lambda)^s p_2$ laying on the boundary of K_2, where p_1 and p_2 are both points in B.

It is a simple consequence of these definitions that every uniformly $S-$convex space is strictly $S-$convex and has got normal structure.
The modulus and characteristic of convexity is also the modulus and characteristic for s_1-convexity. However, for s_2-convexity, things must be adapted acc.. See:

Definition 17. The modulus of s_2-convexity of B is the function $\delta : [0,2] - > [0,2]$ defined by the following formula
$\delta(\epsilon) = \inf \left[2 - ||\frac{x+y}{2^s}|| : x, y \in K_2, \; ||x-y|| \geq \epsilon \right]$.

Remark 1. Trivially, the function $\delta(\epsilon)$ is nonincreasing[2].

3.3 To both:

Definition 18. The characteristic of $S-$convexity of B is the number $\epsilon_0 = \sup[\epsilon : \delta(\epsilon) = 0]$.

4 Extension of main results - $S-$convexity

Theorem 4.1. *B is uniformly $S-$convex then $\epsilon_0 = 0$.*

Proof. Consider B to be uniformly $S-$convex. Then, either for every positive number ϵ, there exists a positive number δ such that for arbitrary $x, y \in K_1$, the inequality

$$||x-y|| \geq \epsilon$$

implies

$$||\frac{x+y}{2}|| \leq 1-\delta(\epsilon),$$

[2]Some typo must have occurred at [1], at this stage.

case B is s_1-convex, or for every positive number ϵ, there exists a positive number δ such that for arbitrary $x, y \in K_2$, the inequality

$$||x-y|| \geq \epsilon$$

implies

$$||\frac{x+y}{2^s}|| < 2-\delta(\epsilon),$$

case B is s_2-convex. In both cases, simple manipulation of the consequences related to the choice of ϵ will lead us to $\delta(\epsilon) \geq \delta$. If $\delta(\epsilon)=0$ then $\delta=0$. Therefore, it must be the case that $\epsilon=0$. □

Theorem 4.2. *If $\epsilon_0 < 1$ then B has got normal structure.*

Proof. The proof is very similar to that for the convex case, small adaptations to the choice of z must be made, however. □

5 Fixed Point Theorems

The paragraph below is almost fully copied from [1]. Regardless of the value of the notion involved, we simply extend their results, or believe we have done so, according to the best of our knowledge.
Let C be a closed and $S-$convex subset of B and let F be a continuous transformation of C into C. By F^n, we denote the nth interaction of F and by I the identity transformation via C.

Theorem 5.1. *If $F^2=I$ and if for arbitrary $x, y \in C$ we have*

$$(*)||Fx-Fy|| \leq k||x-y||,$$

k being a constant such that

$$(**)\frac{k}{2}\left(1-\delta(\frac{2}{k})\right) < 1,$$

then F has at least one fixed point.

Proof. From [1], we take for granted that $||F(\frac{x+Fx}{2})-x|| \leq \frac{k}{2}||x-Fx||$ and $||F(\frac{x+Fx}{2})-Fx|| \leq \frac{k}{2}||x-Fx||$. From both facts, we infer that

$$||\frac{x+Fx}{2}-F(\frac{x+Fx}{2})|| \leq \left(1-\delta(\frac{2}{k})\right)\frac{k^2}{2}||x-Fx||.$$

By putting $G=I+\frac{F}{2}$, we get

$$||G^2x-Gx|| \leq \left(1-\delta(\frac{2}{k})\right)\frac{k^2}{2}||x-Gx||.$$

Hence, the sequence $x_n = G^n x$ is convergent. If $y = \lim x_n$, then $y \in C$ and $y = Gy = Fy$.
All remarks from [1] apply here as well. $\square$

6 Conclusion

We here have managed to re-word a few parts of [1] to suit our taste, have extended their most relevant theorems to $S-$convexity, with the possible exception of the last theorem from [1], regarding Fixed Point, which shall be dealt with in future publications.

7 Bibliography

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