Starants II

M.R.Pinheiro[1]

Abstract

Digressing on the work of Comellas et al. into Deterministic Small-World Networks, and extending the results of Pinheiro's work, we get to mathematically give reasons for the Six Degree of Separation Theory. In this paper, we prove that, in considering people as mathematical objects, along with their circle of acquaintances, there might be, for n people, either a connection so strong as to give 3 degrees of separation, or so weak as to give $2n + (n - 1)$ degrees of separation.
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1 Introduction

Deterministic small-world communication networks were introduced by Comellas, Ozon, and Peters in [COM00]. They are supposed to have strong local clustering (nodes have many neighbors in common), small diameter (largest of the shortest distances between nodes must be small), and would be located between Regular Lattices[2], which are highly clustered, large worlds, where the diameter, or characteristic path length, grows linearly with the number of nodes, and Random Networks, which are poorly clustered, small worlds, where the diameter grows logarithmically with the number of nodes. We shall name them 'medium worlds'.
Circulant graphs are considered part of the deterministic small-world communication networks, once they have strong local clustering. They are included in the class of structured networks.

[1]E-mail: mrpprofessional@yahoo.com. Address: Po Box 12396, A'Beckett st, Melbourne, VIC, AU, 8006.

[2]Obviously, this concept of Lattice must be a more graphical concept, having to do with something else rather than with algebra, once the algebraic concept, well described in [DAH06] does not have much to do with what Comellas writes about at all. On the other hand, [edi06] brings a broader sense, obviously allowed in language, which must be the one referred to by Comellas.

In this paper, we want to contribute to Duncan et al. findings, by establishing a mathematical way of talking about their findings and mathematically proving results about their theory. Our previous results [PIN07] brought constraints and mathematical symbology to Comellas et al. work.

1.1 Notation & some definitions

1. $C_{n,\delta}$ - circulant graph of n nodes and δ (degree) links per node such that each node i is adjacent to nodes $(i \pm 1)$, $(i \pm 2)$, ..., $(i \pm \frac{\delta}{2}(mod\ \ n))$. This graph has got diameter $D = \lceil \frac{n}{\delta} \rceil$ whenever $\delta \neq 2$ and $D = \lfloor \frac{n}{2} \rfloor$ otherwise.

2. Star graph - rooted tree containing n nodes with a central node (root) of degree $(n-1)$.

3. Complete graph on n nodes - graph where every node has got degree $(n-1)$.

4. $S^{C}_{n,\delta}$ - string of n circulant graphs connected by means of K_2 exactly δ times for each circulant graph added (taking away the first and the last graph on the string which will use K_2 exactly $\frac{\delta}{2}$ times to make the connection): each vertex 'i' is connected to $\left(\frac{\delta}{2} - \lambda \pm i\right), 1 \leq \lambda \leq \frac{\delta}{2}$.

5. $C^{C}_{n,\delta}$ - circle of circulant graphs connected by means of K_2 exactly δ times for each graph added: each vertex 'i' is connected to $\left(\frac{\delta}{2} - \lambda \pm i\right), 1 \leq \lambda \leq \frac{\delta}{2}$.

6. S^{C}_{n} - string of n circulant graphs connected by means of K_2 as many times as we like for each circulant graph added.

7. C^{C}_{n} - circle of n circulant graphs connected by means of K_2 as many times as we like for each circulant graph added.

8. $SC_{n,\delta}$ - 'Starant' graph, that is, a circulant graph with degree δ containing a star inside of it whose vertices coincide with the vertices of the circulant graph, a total of n vertices.

9. $S^S_{n,\delta}$ - string of n star-graphs connected by means of K_2 exactly δ times for each star-graph added (taking away the first and the last graph on the string which will use K_2 exactly $\frac{\delta}{2}$ times to make the connection): each vertex 'i' is connected to $\left(\frac{\delta}{2} - \lambda \pm i\right), 1 \leq \lambda \leq \frac{\delta}{2}$.

10. $S^C_{n,\delta}$ - string of n circulant graphs connected by means of K_2 exactly δ times for each circulant graph added (taking away the first and the last graph on the string, which will use K_2 exactly $\frac{\delta}{2}$ times to make the connection): each vertex 'i' is connected to $\left(\frac{\delta}{2} - \lambda \pm i\right), 1 \leq \lambda \leq \frac{\delta}{2}$.

11. $C^{SC}_{n,\delta}$ - circle of n Starant graphs connected by means of K_2 exactly δ times for each Starant graph added: each vertex 'i' is connected to $\left(\frac{\delta}{2} - \lambda \pm i\right), 1 \leq \lambda \leq \frac{\delta}{2}$.

12. $C^S_{n,\delta}$ - circle of stars connected by means of K_2 exactly δ times for each star-graph added: each vertex 'i' will be connected to $\left(\frac{\delta}{2} - \lambda \pm i\right), 1 \leq \lambda \leq \frac{\delta}{2}$.

13. S^S_n - string of n star-graphs connected by means of K_2 as many times as we like for each star-graph added.

14. C^S_n - string of n star-graphs connected by means of K_2 as many times as we like for each star-graph added.

2 Connectivity, Accessibility, Spreadability

We have discussed these terms to exhaustion in a separate piece of work, currently under review for publication as a book chapter. The copy of the first draft of such chapter, however, may be found easily online ($www.geocities.com/mrpprofessional$). Basically, from a mathematical point of view, we believe to have proven that Accessibility is simply not interesting enough, or meaningful. Connectivity, which seems to be subject of Comellas et al., is very interesting, and subtleties will apply for each situation that may fit this concept, such as disease spread. Spreadability is the most interesting concept for our major intent, with this theory, is contributing to stop the spread of diseases, in quickest way as possible, via accurate use of Mathematics/mathematical tools. Having that in mind, we reach our new concepts, exposed from the next section onwards.

3 New results

In [PIN07], we argue that a person's circle of acquaintances might be mathematically represented by means of a Starant graph, which is basically a circulant graph with a star inside of it, making its diameter as short as 2. We now want to consider a situation where we label each vertex of a Starant with a number, and treat the connections as cycles. Bearing in mind that, no matter how many people a person knows, one will always get the same diameter in a Starant, 2, makes us think that it is easy to mathematically prove Duncan's theory. One must be very attentive, though, for the fact that a person's circle of acquaintances might be an 'incomplete' Starant, that is, it is still a star, that is for sure, but the vertices of the star might not have edges between them, that is, a person might know n people who actually do not know each other.
In this sense, the theory we present here is still not complete. But it is already a good start.
Another issue that should deserve our careful observation is that, in considering people's circle of acquaintances to be our Starants, it does not make any sense connecting the root of the star graph, that is, the person who originated that circle of acquaintances, to another root. It is very logical to understand that if I have m people in my circle of acquaintances, all of them are going to be part of my Starant. Once we are not working with the dynamics of the problem, just the static, it only makes sense to think of possible connections from the people that we know to other people that someone else knows, never from root to root. That will always establish one degree of separation inside of each Starant (at least one mediator, that is, at least 2 degrees between each couple of roots). Therefore, for a square of n people, there are at least n degrees of separation (actually, $2n$).
Right here, one starts doubting Duncan's statements immediately. But if one thinks of the folding of the universe, similar to the Möbius band [gC06], and squeezing, then one may see that as possible, that is, in an infinite grid of people, and considering one person on the top and another on the bottom part of the Möbius band, then it is all possible. What we are actually stating

is obviously that his theory would be a dream, only achievable eventually, if the right people in the universe are picked in an infinite number of participants in what regards connectivity, not accessibility. It is also obvious that, as much as the concept used by Comellas on Lattices differs from the mathematical one, our concept of connectivity differs from the usual media concept. Given the limitation of language, in comparison to the speed of thought [PIN06a], that was the best term we could come up with, so that we beg the reader to read our definition of connectivity, recursively, each time the word is applied in our papers.

For the good understander, we have just stated, and proved, that it is humanly impossible that Duncan's theory works for any amount of people, even infinite, considering the way static chains work. Well, in Language, it is all acceptable, natural language, as we explain in [PIN06a]. But, in Science, that is, Philosophy, which encompasses Mathematics, it is not possible to accept his assertion as true, that is, just like in our Sorites solution from [PIN06b], we are forced to say 'no' to the veracity of his 'findings'.

Once any probabilistic chain may be photographed from a static point of view, his theory makes no sense at all in terms of connectivity. May still make sense in terms of accessibility, however.

The reason behind that is, obviously, that, irrespective of the length involved in the chain, whatever it is, even supposing the Möbius band fits the picture of the chain as illustration, to make it easier, telecommunication could always create regular lines through which anyone could be reached via two steps (place an ideal, abstract, communication hub in the middle of the Möbius band, as we did with the middle of our Circulant in order to create our Starant, and you can easily see how there could be a star generated by that hub, reaching every single person in our obviously finite chain of World people - only infinite in terms of human counting: perspective, referential, like in Physics; infinite for the human eyes, human life time, in terms of inability of counting one by one. As for the infinity of the population, not even so, in modern times, given the Statistics advances, already in place, which allow us, via a single Internet click, to learn the World population counting, with little mistake).

Suppose, then, that we have a labeled Starant,$SC_{5,1}$, with the vertices from the Circulant labeled with numbers from one to five and the root of the star labeled with the number 6. For this graph, it is true that the following cycles exist:

$$(12)(23)(34)(45)(15)(64)(65)(61)(62)(63).$$

Suppose, now, that we have another Starant, also $SC_{5,1}$, which we connect with the previous one, let's name it $SC2_{5,1}$. We now label its vertices from 7 to 12, 12 being the root of the star. For $SC2_{5,1}$ we then get:

$$(78)(89)(9\ \ 10)(10\ \ 11)(7\ \ 11)(12\ \ 9)(12\ \ 8)(12\ \ 7)(12\ \ 10)(12\ \ 11)$$

Suppose now that $SC_{5,1}$ connects to $SC2_{5,1}$ via K_2, originating a new cycle: $(10\ \ 1)$. Therefore, the longest of the shortest paths between them (diameter) is now the diameter of $SC_{5,1}$, 2, plus the diameter of $SC2_{5,1}$, 2, plus the edge connecting them, that is 5. One must bear in mind that this actually means that there is no intersection between the two circle of acquaintances, and we might represent this by means of the following symbology:

$$SC_{5,1} \bigcap SC2_{5,1} = \varnothing.$$

We know that one of the ways of decreasing this diameter is decreasing one of the Starants diameter, that is, for instance, making their degree 1, instead of 2. Notice that this is possible, in real life, because it is possible that all our acquaintances know each other, even in the same level of connectivity. Therefore, if we do this to one of them, we then get diameter 4 to our set of two Starants. If we do this to both of them, we get diameter 3. This proves that the connection of a Starant to another takes something between 3 and 5 steps, that is, the total diameter of an 'Australian square', let's call it $D_{A_{n,\delta}}$ may be expressed as

$$3 \leq D_{A_{n,\delta}} \leq 5.$$

With this, we write our first theorem:

Theorem 3.1. *An Australian square has got diameter between* 3 *and* 5*.*

Proof. As written above. □

which follows our first new definition:

Definition 4. *We call an Australian square the static picture of two Starant graphs, with no intersection between their set of vertices, connected to each other via a mandatory K_2.*

The importance of giving a name to our set of starants, glued this way, is making it possible to refer to precisely that static picture, with no mistake.

Once there is no similar graph theory object like that, given Starants did not exist before either, we must create a new name. Having into sight that Australia is first world, but not as first World as the States, being a few degrees away in terms of development of Democracy, and everything else, we actually think it is all real-life-related coherently. The best intentions of giving popular names to mathematical, or computer objects, is making them easy to remember in the heads of non-mathematicians. We believe that this is the best choice we could possibly have. Once we consider that there is regularity, also expected in development (balance of things) it should be a highly developed society, not a messy one, where relationships are highly uneven when one member of society is compared to another. If indices are correctly built, and it is not our area of study, so that we just assume they are, first World nations should have their name originating from the fact that they are much more developed than the others, and because we have associated development with accessibility, and we are interested in reaching Duncan's work, which we already stated to be only possibly valid in the scope of accessibility, that is severely relevant. With this in mind, one may see that, if we duplicate the amount of Australian squares, that is, we make them two, in getting at least 3, and at most 5, as diameter for each Australian square, and supposing they only connect either horizontally, or vertically, our new diameter would be between 5 and 11.
So far, notice the fact that we only allow our starants to connect from end of star to end of star via a K_2 graph, with no exception. We seriously believe this would be the minimum that would happen in any society, or randomly chosen group of people, in a static picture.
If we allow them to connect diagonally, though, say that only $SC2_{5,1}$ and $SC4_{5,1}$ do it, we still get no reduction. However, if we also diagonally connect $SC1_{5,1}$ and $SC3_{5,1}$, we then get a reduction of diameter from the bounds 5 and 11 to 3 and 5. With this, we have proved that:

Theorem 4.1. *An American square has got diameter between* 3 *and* 11*.*

And we now introduce a new definition:

Definition 5. *We call American square the set of two Australian squares, with no intersection between their sets of vertices, connected by means of at most one K_2-sort-of-connection between each couple.*

Notice the danger here: If this sort of reasoning spreads, then Duncan not only is correct but it would actually be 3, half of his predictions, rather than

6, once it all is passive of reduction to 3, sufficing that diagonal connections are made by at least one member of each person's circle of acquaintances. However, here, we must notice that this regularity would have to be possible to be kept for an extenuating amount of people, that is, the World population (what is, again, impossible). Notice that it is apparently always necessary that each circle of acquaintances have at least one member who knows someone from each other circle belonging to the description of the actual World. Considering the dimension of things, one may see that it would be easily achievable in small populations, but not in large ones, the opposite of his statements. In this case, the hub would be replaced by diagonal connections transferred to the already existing graphs.

One may notice that the higher bound is always growing and the lower bound is always the same. We already wrote in detail about the lower bound and do think we have exhausted its possibilities. On the other hand, the higher bound is never interesting, once, for disease spread, we are always worried about the lower bound, and it could also be useful for communication theories. In this case, it has become easy to predict, by means of mapping the territory, with our American, or Australian squares, speed of spread.

We may, then, state a more generic theorem:

Theorem 5.1. *For a group of n people*[3] *(never containing any fully isolated member), it takes something between* 3 *and* $2n + (n - 1)$ *steps for them to communicate amongst themselves*[4] *if and only if the actual graph connections correspond to all possibilities of connections involved, that is, if telecommunication is ever considered, it has to be confined to the boundaries of our graphical display.*

Therefore, if we have 100 people, it might take 299 steps to get effective communication/transmission of disease.

[3] Remember, at this stage, that whenever we, human beings, refer to any amount of people, we will never include their circles of acquaintances, but such must be counted, so that our theory is complete.

[4] Considering they must go through a possible path in the graph representing their connectivity. This reasoning actually fits precisely the scope of disease spread and that is where we disagree with Duncan's choice about communication. We are talking about face to face contact that is actually possible for connection reasons of any sort.

6 What are Australian/American squares good for?

Notice that an Australian square holds similar figures, in what regards steps in communication, to those of the six degree separation theory. That must mean there is a way of corresponding their theory with our graphs, in order to explain it all mathematically. If not precisely to our graphs, to something very similar, bearing small modifications. What that means is that we can also prove it geometrically, what makes the explanation not only scientific, but also popular.
Any mathematical theory is proven wrong by a single example where it does not work. Duncan's theory is in the scope of probability, however, and that is when it becomes really hard to prove that it is wrong, once it is supposed to work in terms of average only, so that there might be a case, or two, where that fails but, in average, it all works fine, making the theory true as a whole. However, if we find a single person from whom it is impossible to find any connection with the Namibia, or Afheganistan, president, for instance, there is then a distance of infinity to that president, and there is no average of 6 made ever possible...
It is obviously possible to find that example, however. Take the simplest one: a retard, a person who never goes out, only knows their nurse, who knows only their father, or mother, for instance, who are humble people, and have never traveled. They all live in a community where no airplanes, no buses, no nothing gets there - an island, yet not found by anyone else. Now we have a whole community stuck in the past somewhere in this World. This way, the degree is infinity, and the average is then infinity, and his theory can only be wrong. With the Mobius band argumentation, we took the assertion that the degree was 6, not as average, but as usual, into consideration. And we are now talking about the average, which is the actual proposal of Duncan's work. In any hypothesis, it is not true.
We have proved, however, that, with highly developed communication, any communications network could have a 3 degree diameter. In this hypothesis, stopping disease, in the quickest time as possible, would mean two steps from everyone else's first contact. That would be the only way of guaranteeing effective disease spread stop for any disease.
We believe now to have settled the matter, in terms of showing to everyone else that six degree theories could only refer to accessibility because it is

mathematically, statistically, and sociologically, unacceptable otherwise by human reasoning.
Therefore, the importance of our work is also the same as in the Sorites paradox: defining things precisely, their scope. Not only that is a basic step for further progress towards the right direction of research, but helps everyone else to see the value of our/their research into small/medium worlds.
In that sense, if there is any doubt about the use of our terminology, we hope to have finally sorted those doubts out.
Besides, it seems that, in real life, it is quite possible that someone we know is part of someone else's circle of acquaintances. This situation would originate, for instance, an Australian square with acquaintances' circle intersection different from the empty set. This is a case where the vertices of the Starants will overlap. With this, there is one less step to go from one Starant to the other and we are stuck with $2n$ distance in the string situation but 3 as minimum as well[5]. Easy to see that this is the only way we could possibly get 6, that is, Duncan assumes there is always, either in majority, or in average, one acquaintance in common between two different people's circle of acquaintances.
Another interesting point is that if we depart from static small pictures, like we have done, it is usually the case that total strangers never have common acquaintances and we actually believe this is the point in Duncan's theory presentation. As failure as the Sorites paradox, its presentation illudes us to think Duncan intends a message that he never did, perhaps, once if he had ever intended the message we get immediately, given the illusion, he would always be incorrect. The phenomenon just described happened to the Sorites, and applying same sort of reasoning to this problem, of networks, will lead to:

- Presenter: Someone in the audience, please, raise your hand... What is your name?
- Blah, blah, blah.
- P: Mrs. Blah, I want you to contact the President of the USA now. Can you think of how?

[5] 2 degrees might occur, but occur if and only if two different people hold precisely same circle of acquaintances and these two people do not know each other. A bit too exotic for real life, but possible.

- Mrs. Blah goes: well, I know someone who knows someone...
- P: Call!
- (Mrs Blah calls that someone and... wow, in 5 calls we get to the President!)
- P: You see, anyone, anywhere, may get to anyone else, anywhere, in at most 6 communication steps!

Just like our Sorites paradox, we get it all wrong this way...(please check on [PIN06b]). First of all, Duncan never said that at all. He said that, in average..., which means not everyone, not at any time...(it might even be the case that Mrs Blah never gets to speak to the President at all - however, notice the failure if we take this one in, infinity for the average calculation is back, the killer...). And there is another interesting point: Even if I know someone in the President's office, it might be the case that the telephone, even with me having his/her personal number, is never answered by the time we need, in the presentation or test. If that happens with everyone else, once it is obviously impossible that it all is about personal contact, where it obviously fails, or the majority of people involved, his communication theory demands a time extension to be correct. But that time makes the problem not being solved by the time which is necessary to prove his assertion, so that it might not be true even if accessibility is of 2-steps type, that is, best as possible, top Democracy in the World: It suffices having a problem with the telephone company, at that time, and date, and it will never be true (remember what 'never' here means...killer!).
It will never be true in average, it will never be true in any possible sense, then!
Boring enough, the static picture of Duncan's problem might as well be the one where the telephone lines all fail. In this case, none of our theories apply. Basically, we write about possible connections as well. And, we did mention face to face contact. That is why we worried about reducing the scope of the population involved, even because if we can control every small village, in terms of disease spread, quickly, via their local medical center, the epidemics will, indeed, be stopped.
With all these conclusions, we actually notice that the only interesting contribution that Mathematics could provide the World with, using Duncan's

work, would be in the area of very small graphs, that is, describing determined finite number of people involved. We obviously consider Community Health Centers to be the societies involved, with their usual patients (what is modernly labeled as family care centers). Once more, for it all to work and being suitable for Mathematics contribution, it would be necessary the frequency, and the obligatory attachment, of all that population to their local community health center. However, the Mathematics, as usual, may be there, ready to be applied in ideal situations, which will, perhaps, never happen in real life or, only by luck, will be found.
Therefore, Duncan's theory seems to be quite attractive, in the same sense the Sorites paradox is: something with wrong presentation, but generating insights that may solve other problems.
The Sorites made us come up with really useful insights for Logic and other sciences. The same way, we may start thinking of these other concepts brought up by Duncan's theory, in the vain trial of fitting some Maths where there is no room for it: Human Communication. Statistics is suitable because it is about average - actual facts - and it may account for changes (it is not a static picture in a determined instant of time). However, cannot ever be of mathematical use. We truly think the only possible contribution of Duncan's theory, for something passive of Maths, is what we have done: the graphs, so that computers might be used to control, predict, and register, disease spread, in a more accurate way. Anything else falls inside of the same scope, reported in the Sorites problem, and is useless. The creation of a new graph, however, gives us other paths to pursue, in terms of purely mathematical theories.

7 Conclusion

In this paper, we have worked with new mathematical objects using the concepts of cycles and Starants. We have introduced both Australian squares and American squares. Our work seems to be, in a sense, on the back of Duncan's et al. reasoning, but he never stated so or hinted us on that. Therefore, we take it as ours, 100%. Our conclusions are suitable to prove, control, measure, and study disease spread scientifically/computer-friendly in a way to provide actual solutions - practical solutions, rather than just useless studies, as the ones presented in [NEW02], which might be totally accurate but represent no practical contribution to the computer environment so that they

are not applicable in terms of providing governments with actual tools to monitor, control, stop, disease spread. Our work gives the computer scientists a way of inserting data in the computers accurately so that any amateur may make use of it and study/solve their problems of disease spread besides cataloguing. And we also believe to have achieved the best model for such: a totally perfect model of reality via graph theory. With the improvement of our mathematical tools, we wish to make our graphs computer-friendly, in a way to make it easy for Statistics to forecast immediately, in real time (a single Internet click away?).
Obviously, this work supplements any studies of the sort presented in [NEW02]. However, if one takes the spread to be the quickest as possible, using their actual spreading function/mathematical model, diseases will always be stopped the quickest way. For that end, it suffices having a model for the worst case scenario (quickest).
We also believe we have progressed further in terms of providing restrictions for the situations in which Duncan's theories apply. We have proved that Duncan may only be referring to Accessibility, as defined in our terms (not Spreadability, or Connectivity). We have also proved that he has to be scientifically wrong in his most famous assertion.
We have proved that his theory must be very well restricted in order for it to work.
We seem to also have empowered people with 'static power'prediction over a certain period time, of precise figures, in terms of Connectivity of people, if we are right about the way they connect, that is, given hour X of day Y, we are able to mathematically measure Connectivity, or predict it, in terms of what is possible to have in human's face to face frequent contact.

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