Representation of Signals and Systems

- Fourier Analysis
- Bandwidth
- Hilbert Transform
- Complex Representation of Signals and System

Fourier Analysis (1/7)

- 定義:g(t)為nonperiodic deterministic signal
  - Fourier transform $G(f)=\int_{-\infty}^{\infty} g(t)\exp(-j2\pi ft)dt$
  - 反之，已知G(f)則inverse Fourier transform

$$g(t)=\int_{-\infty}^{\infty} G(f)\exp(j2\pi ft)df = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\exp(j\omega t)d\omega$$

  - Transformable
    - Dirichlet's conditions (充分非必要)
      - 單一值，有限區間最大最小值及不連續有限個
      - 絕對可積分 $\int_{-\infty}^{\infty} |g(t)|\,dt<\infty$
    - 實際訊號；energy signals $\int_{-\infty}^{\infty} |g(t)|^2\,dt<\infty$
  - Fourier transform=Spectrum
    - |G(f)| magnitude spectrum, {G(f)} phase spectrum

Fourier Analysis (2/7)

❑Properties of the Fourier transform

1. Linearity	$ag_1(t) + bg_2(t) \rightleftharpoons aG_1(f) + bG_2(f)$ where a and b are constants
2. Time scaling	$g(at) \rightleftharpoons \frac{1}{\lvert a \rvert} G\left(\frac{f}{a}\right)$ where a is a constant
3. Duality	If $g(t) \rightleftharpoons G(f)$, then $G(t) \rightleftharpoons g(-f)$
4. Time shifting	$g(t - t_0) \rightleftharpoons G(f) \exp(-j2\pi f t_0)$
5. Frequency shifting	$\exp(j2\pi f_c t) g(t) \rightleftharpoons G(f - f_c)$
6. Area under $g(t)$	$\int_{-\infty}^{\infty} g(t)\, dt = G(0)$

Fourier Analysis (3/7)

7. Area under $G(f)$	$g(0) = \int_{-\infty}^{\infty} G(f)\, df$
8. Differentiation in the time domain	$\frac{d}{dt} g(t) \rightleftharpoons j2\pi f G(f)$
9. Integration in the time domain	$\int_{-\infty}^{t} g(\tau)\, d\tau \rightleftharpoons \frac{1}{j2\pi f} G(f) + \frac{G(0)}{2} \delta(f)$
10. Conjugate functions	If $g(t) \rightleftharpoons G(f)$, then $g^*(t) \rightleftharpoons G^*(-f)$
11. Multiplication in the time domain	$g_1(t) g_2(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda) G_2(f - \lambda)\, d\lambda$
12. Convolution in the time domain	$\int_{-\infty}^{\infty} g_1(\tau) g_2(t - \tau)\, d\tau \rightleftharpoons G_1(f) G_2(f)$

Fourier Analysis (4/7)

❑ Dirac delta function

- 延伸transform至不符Dirichlet conditions
  - ☞結合Fourier series及Fourier transform
  - ☞使適用power signals
- Dirac delta function or unit impluse: 為even fun.

$$\delta(t)=0, t\neq 0 \text{ and } \int_{-\infty}^{\infty}\delta(t)dt=1$$

- Shift property $$\int_{-\infty}^{\infty} g(t)\delta(t-t_0)dt = g(t_0)$$
- Replication property $$\int_{-\infty}^{\infty} g(\tau)\delta(t-\tau)d\tau = g(t)$$

  t→τ, t_0→t, δ(τ -t)= δ(t- τ)
- 非實際，可看成單位面積pulse的limiting form

Fourier Analysis (5/7)

❑ Fourier Transforms of periodic signals

- 週期函數complex exponential Fourier series
  - ☞ $$g_{T_0}(t)=\sum_{n=-\infty}^{\infty} c_n \exp(j2\pi n f_0 t)$$
  - ☞Complex Fourier coeffiecient

    $$c_n=\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_{T_0}(t)\exp(-j2\pi n f_0 t)dt$$
  - ☞基頻f_0與週期T_0：$f_0=1/T_0$
- 表為傳立葉轉換
  - ☞ $$g(t)=\begin{cases} g_{T_0}(t) & -T_0/2<t\leq T_0/2 \\ 0 & elsewhere \end{cases} \text{ then } g_{T_0}(t)=\sum_{m=-\infty}^{\infty} g(t-mT_0)$$

Fourier Analysis (6/7)

☞係數 $c_n = f_0 \int_{-\infty(-T/2)}^{\infty(T/2)} g(t)\exp(-j2\pi nf_0 t)dt = f_0 G(nf_0)$

then $g_{T_0}(t) = \sum_{m=-\infty}^{\infty} g(t-mT_0) = f_0 \sum_{n=-\infty}^{\infty} G(nf_0)\exp(j2\pi nf_0 t)$

❁頻譜說明

☞週期函數中一個週期之轉換為連續且無限頻寬

☞週期函數轉換則為離散且分佈在基頻的整數倍

❑Fourier-transform pairs: A6.3

❑Transmission of signals through linear systems

❁System: input (excitation)產生output (response)

☞Linear system: 具principle of superposition

✐時域之脈衝響應h(t): zero initial condition, input=δ(t)

✶ Time invariant則response與input時間無關

Fourier Analysis (7/7)

✐線性非時變之激發x(t)與響應y(t): convolution integral

✶ Commutative: $y(t) = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

✶ 時間關係：激發時間τ，響應時間t，系統記憶時間t-τ

✶ 某時間的輸出為輸入過去歷史之加權積分，加權值即脈衝響應(當作記憶函數) $\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$

❑Frequency response of linear time-invariant system

❁頻寬響應H(f)=F[h(t)]

☞輸入x(t)=exp(j2πft),輸出y(t)=H(f)exp(j2πft)

✐單頻輸入，則只在單頻放大輸出，放大率為該頻之H(f)

☞複數表示法H(f)=|H(f)|exp[jβ(f)], |H(f)| magnitude response, β(f) phase (response)

✐若h(t)為實數，則H(f)為conjugate symmetry，即|H(f)|為偶函數 |H(f)|=|H(-f)| ， β(f)為奇函數β(f)=-β(-f)

✐極座標表示log H(f)=log|H(f)|+jβ(f) =α(f)+jβ(f)

✐α(f)為gain，單位為neper, β(f)單位為徑度

✐Neper vs. dB: α'(f)=20 $\log_{10}$|H(f)|=20 log|H(f)|/2.3=8.69 α(f)

Bandwidth (1/4)

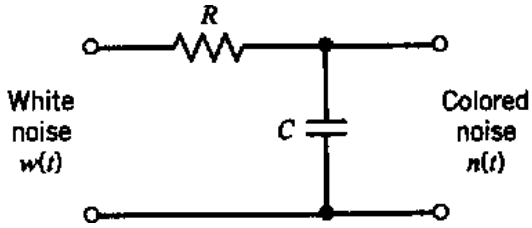
- 一些定義
  - 訊號時域及頻域的描述是inversely related
    - 只能指定時間函數及頻譜其中一項，不可兩項同時
    - 一邊有限則另一邊無限(雖會漸小)，sinc(x)即為一例
  - 頻寬：有意義頻譜內容的範圍中正頻率的部分
    - sinc(2Wt)頻寬為W ($\frac{1}{2W}\text{rect}(\frac{f}{2W})$, well defined)
    - 看Main lobe (null)
      - Low-pass:主葉一半，例方塊波時間長度T,主葉2/T, 頻寬1/T
      - Band-pass:主葉以f_c為中心, f_c夠大則頻寬即主葉寬此即null-to-null bandwidth

Bandwidth (2/4)

- 3-dB bandwidth $20\log_{10}\frac{1}{\sqrt{2}} = 10(\log_{10}1 - \log_{10}2) = 10\times(-0.301) = -3$
  - 低通，頻率為0時最高，則振幅頻譜在正頻率部分降至最高的$1/\sqrt{2}$時
  - 帶通，頻率f_c時最高，則振幅頻譜在正頻率部分降至最高的$1/\sqrt{2}$時的兩頻率差
  - 優缺點：優點可在圖(以dB表示)上直接畫出；缺點是當尾部下降很慢，則易有誤解(以為它頻寬很寬)
- Root mean square (rms) bandwidth (for low pass)
  - Normalize成機率分佈 $\frac{|G(f)|^2}{\int_{-\infty}^{\infty}|G(f)|^2 df} (= p(f))$
  - $W_{rms} = \sqrt{E(f^2)}$ (A2.25)
  - 優缺點:數學上好處理，但實際上不好量

Bandwidth (3/4)

- ❑Time-bandwidth (bandwidth-duration) product
  - Pulse signals的duration與bandwidth的乘積為常數
    - 可由time-scaling property of Fourier transform看出
    - 改變頻寬定義只會影響常數值，但關係不變
- ❑Noise equivalent bandwidth
  - 低通濾波器，White noise with zero mean power spectral density $N_0/2$，則輸出雜訊功率$N_0/(4RC)$，half-power (3-dB) bandwidth $1/(2\pi RC)$

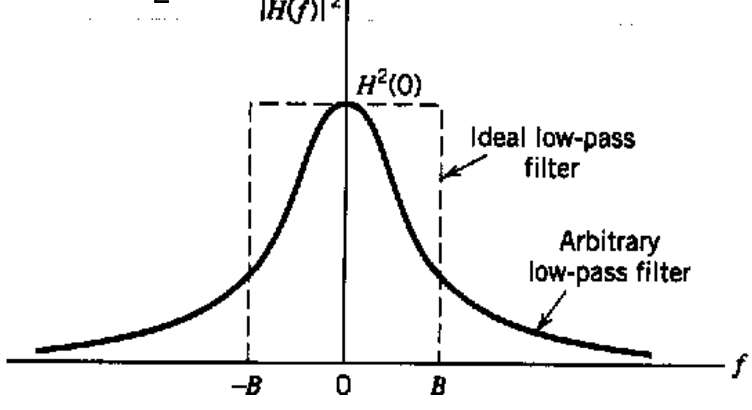


Bandwidth (4/4)

- 低通濾波器，White noise with zero mean power spectral density $N_0/2$，則 $B = \frac{\int_0^\infty |H(f)|^2 df}{H^2(0)}$

$$N_{out} = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df = N_0 \int_0^\infty |H(f)|^2 df$$

$$N_{out} = \frac{N_0}{2}(2B)H^2(0) = N_0 B H^2(0)$$



Hilbert Transform (1/2)

- 緣由、定義、說明
  - Fourier用在frequency-selective, Hilbert用在phase-selective
  - Hilbert transform $\hat{g}(t)$將訊號轉±90º
    - ☞ $\hat{g}(t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(\tau)}{t-\tau}d\tau=g(t)*\frac{1}{\pi t}$
    - ☞Inverse Hilbert transform $g(t)=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\hat{g}(\tau)}{t-\tau}d\tau=\hat{g}(t)*\frac{-1}{\pi t}$
    - ☞Hilbert-transform pair: Table A6.4
  - 可視為g(t)與1/πt之convolution
    - ☞F(1/πt)= - j sgn(f), signum function $\text{sgn}(f)=\begin{cases}1, & f>0\\ 0, & f=0\\ -1, & f<0\end{cases}$

Hilbert Transform (2/2)

- ☞Fourier transform $\hat{G}(f)=-j\,\text{sgn}(f)G(f)$
- ☞Hilbert transformer: 將原訊號正頻率部分位移負90度，負頻率部分位移正90度，但大小不變

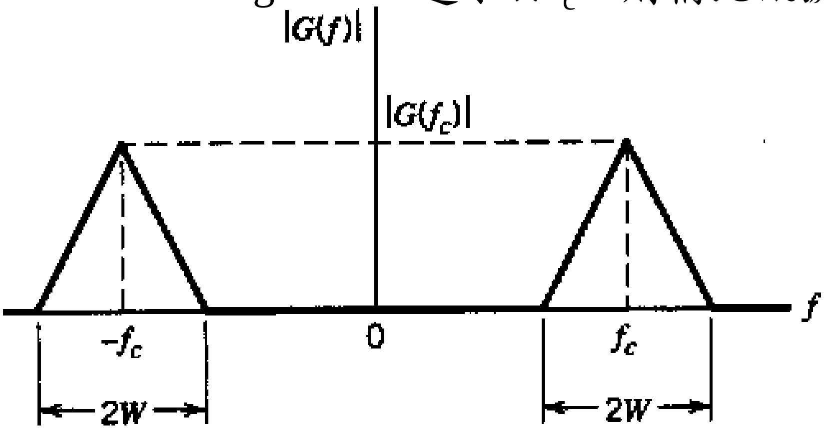
- Properties of the Hilbert transform
  - 與Fourier不同，Hilbert transform皆在時域
  - 假設g(t)為實數
    - ☞轉換前後之頻譜大小相同
    - ☞轉兩次則得-g(t) (因為不管正負頻率皆轉180度)
    - ☞Orthogonal: $\int_{-\infty}^{\infty}g(t)\hat{g}(t)dt=0$

Complex Representation of Signals and System (1/10)

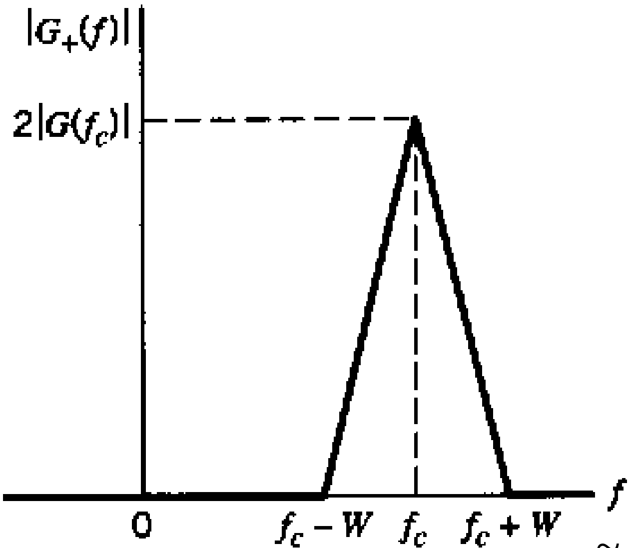
- ❑Pre-envelope (analytic signal) $g_+(t) = g(t) + j\hat{g}(t)$
  - ❁Fourier transform
    - ☞負頻無值 $G_+(f) = G(f) + \mathrm{sgn}(f)G(f) = \begin{cases} 2G(f), & f > 0 \\ G(0), & f = 0 \\ 0, & f < 0 \end{cases}$
    - ☞零頻通常為零，因帶通
    - ☞正頻率有值為原訊號之兩倍
  - ❁由g(t)決定$g_+(t)$的兩種方式
    - ☞先求Hilbert transform，再求$g_+(t)$
    - ☞先求G(f)，再求$G_+(f)$ ，反轉回$g_+(t)$
  - ❁負頻率之pre-envelope $g_-(t) = g(t) - j\hat{g}(t) = g_+^*(t)$
    - ☞只在負頻率有值 (A2.41)
    - ☞$g_+(t)+g_-(t)=2g(t)$

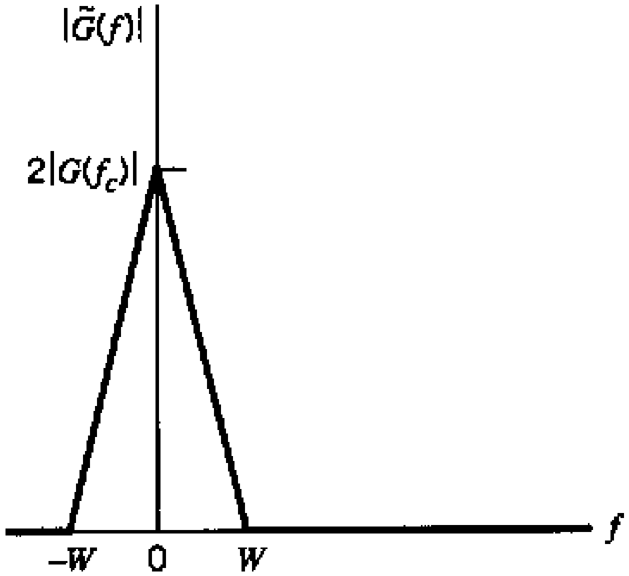
Complex Representation of Signals and System (2/10)

- ❑Canonical representations of band-pass signals
  - ❁一些名詞定義
    - ☞Band-pass signal (2W): 在2W以外之訊號可忽略
    - ☞Carrier frequency f_c:帶通訊號之中心頻率$\pm f_c$
    - ☞Narrowband signal: 2W遠小於f_c，則稱此訊號為窄頻



Complex Representation of Signals and System (3/10)

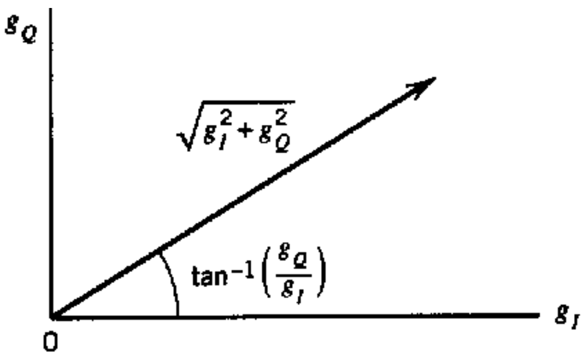


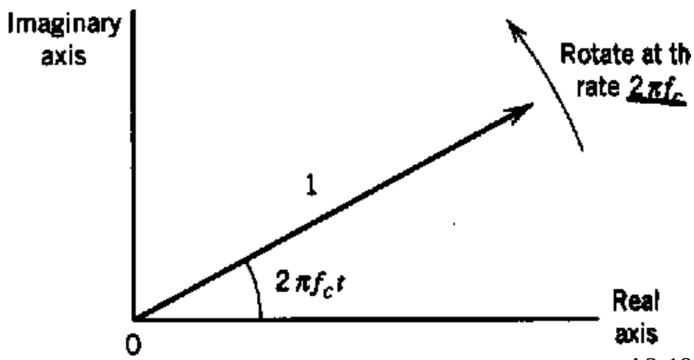


- $g_+(t)$表為$g_+(t) = \tilde{g}(t)\exp(j2\pi f_c t)$
  - $\tilde{g}(t)$為其complex envelope，依Fourier頻率位移特性，其為低通訊號

Complex Representation of Signals and System (4/10)

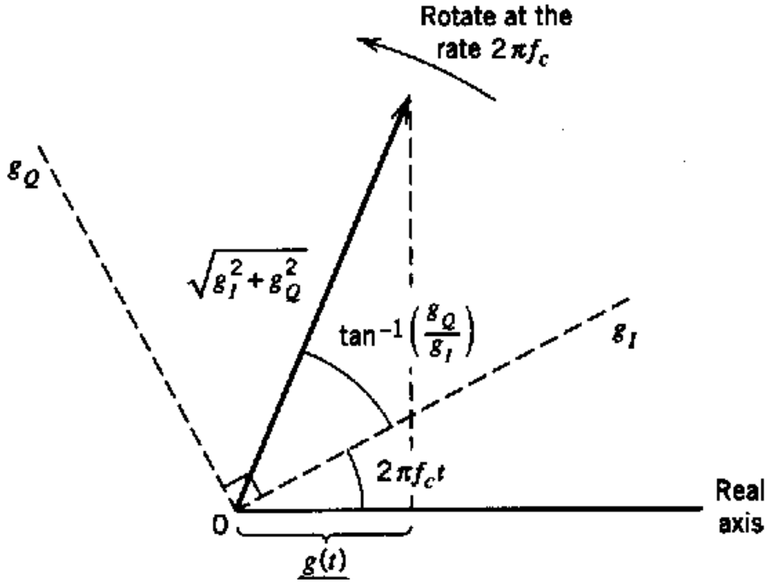
- 原訊號表為complex envelope及canonical form
  - $g(t) = \mathrm{Re}[g_+(t)] = \mathrm{Re}[\tilde{g}(t)\exp(j2\pi f_c t)]$
  - Cartesian form $\tilde{g}(t) = g_I(t) + jg_Q(t)$, $g_I(t)$及$g_Q(t)$皆為低通
    - Canonical (standard) form $g(t) = g_I(t)\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t)$
      - $g_I(t)$ in-phase component, $g_Q(t)$ quadrature component
- 圖示說明: Phasors相乘則角度相加、長度相乘





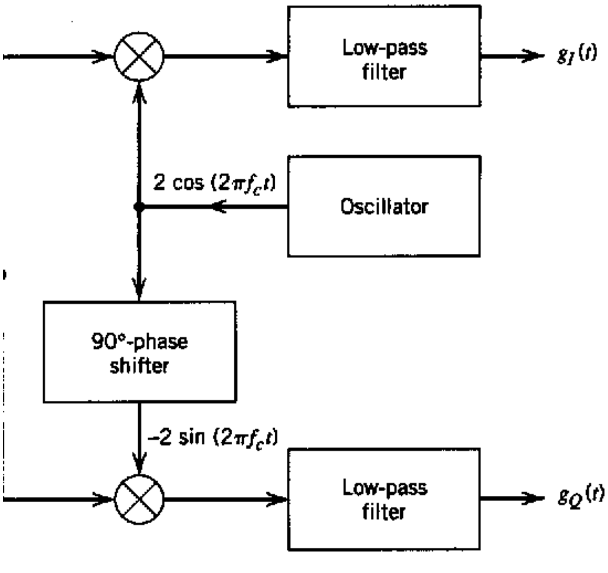
Complex Representation of Signals and System (5/10)

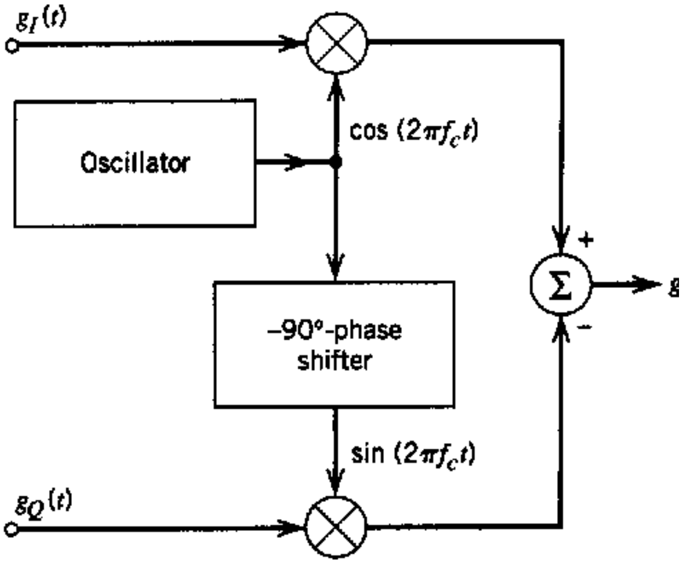
☞$g_I(t)$-$g_Q(t)$平面以2f角速度旋轉
☞實部為橫軸之投影



Complex Representation of Signals and System (6/10)

❀$g_I(t)$、$g_Q(t)$及g(t)之產生





Complex Representation of Signals and System (7/10)

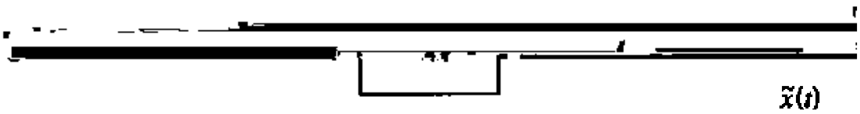
- Polar form $\tilde{g}(t) = a(t)\exp[j\phi(t)]$
  - a(t)及φ(t)皆為實數低通
  - $g(t) = \text{Re}[a(t)\exp(j2\pi f_c t + j\phi(t))] = a(t)\cos[2\pi f_c t + \phi(t)]$
    hybrid form of AM and angle modulation
  - a(t)為g(t)之(natural) envelope，φ(t)為其phase
- 總結
  - Band-pass (modulated) signal可表為
    - In-phase及quadrature components
    - Envelope及phase
  - 所有資訊內容完全保留在complex envelope

Complex Representation of Signals and System (8/10)

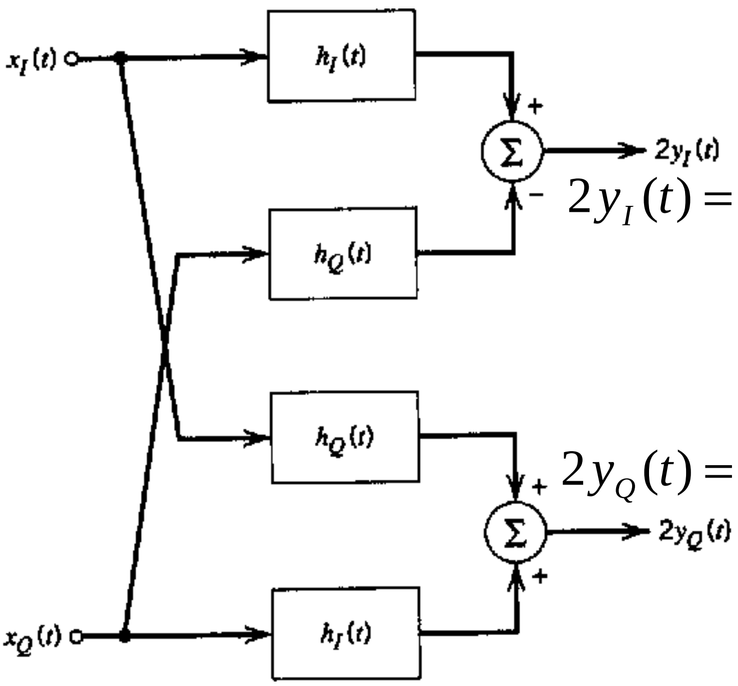
- Terminology (三種envelopes之說明)
  - Pre-envelope $g_+(t)$:訊號正頻率部分(之兩倍)
  - Complex envelope $\tilde{g}(t)$: Pre-envelope位移至基頻
  - Envelope: $a(t) = |\tilde{g}(t)| = |g_+(t)| \neq |g(t)|$
  - Real or complex, low or band pass (另有整理)
    - $g_+(t)$: complex, band-pass, depends on f_c
    - $\tilde{g}(t)$: complex (generally), low-pass
    - a(t): real, low-pass
  - 各關係式 (另有整理)

Complex Representation of Signals and System (9/10)

- Band-pass systems
  - Band-pass訊號不好分析，所以分析其同形物
  - Narrowband x(t)，線性非時變帶通系統之脈衝響應為h(t)，皆可表為IQ form或複數取實部
  - $\tilde{h}(t)$可由$h_I(t)$、$h_Q(t)$求得或H(f)求得(另述)
  - 輸出 $y(t) = \mathrm{Re}[\tilde{y}(t)\exp(j2\pi f_c t)]$
    - ☞ $2\tilde{y}(t) = \tilde{h}(t) * \tilde{x}(t)$



Complex Representation of Signals and System (10/10)



$$2y_I(t) = h_I(t) * x_I(t) - h_Q(t) * x_Q(t)$$

$$2y_Q(t) = h_Q(t) * x_I(t) + h_I(t) * x_Q(t)$$