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Possibilistic linear programming: a brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem

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Abstract

In this paper, we review some fuzzy linear programming methods and techniques from a practical point of view. In the first part, the general history and the approach of fuzzy mathematical programming are introduced. Using a numerical example, some models of fuzzy linear programming are described. In the second part of the paper, fuzzy mathematical programming approaches are compared to stochastic programming ones. The advantages and disadvantages of fuzzy mathematical programming approaches are exemplified in the setting of an optimal portfolio selection problem. Finally, some newly developed ideas and techniques in fuzzy mathematical programming are briefly reviewed. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Fuzzy mathematical programming; Fuzzy constraint; Fuzzy goal; Possibility measure; Necessity measure; Simplex method; Stochastic programming; Portfolio selection

1. Introduction

The notion of fuzzy set is widely spread to various fields after a resounding success in the applications of fuzzy logic controllers in late 1980s. The application to mathematical programming has relatively long history (see [107]). In spite of the fact that there is no big boom in applications of fuzzy sets theory to the mathematical programming, the history of fuzzy mathematical programming is rich enough. This is the fruit of the continuous efforts of the researchers in that topic. Therefore, it is not easy to describe all of the fuzzy mathematical programming techniques in one paper.

In this paper, we restrict ourselves to describing the essence of fuzzy mathematical programming, especially possibilistic linear programming and to demonstrating its characteristics by using concrete examples, instead of introducing a lot of fuzzy mathematical programming techniques. The readers who are interested in various fuzzy mathematical programming techniques are referred to Slowinski [94], Luhandjula [60], Inuiguchi et al. [33], Rommelfanger [89], Sakawa [92] and Lai-Hwang [56,57].

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In the first part of the paper, we introduce an illustrative realistic example in order to explain why the fuzzy mathematical programming problem is developed. Here it is emphasized that two kinds of uncertainty, ambiguity and vagueness are treated in the fuzzy mathematical programming. A general fuzzy mathematical programming approach is described. After this general description, a fuzzy mathematical programming technique is applied to a concrete realistic example in the succeeding sections. In this part, the required knowledge for developing the method is also explained; moreover, in order to emboss the characteristics of the fuzzy mathematical programming approach, the difference from the conventional mathematical programming approach is examined. In the second part of the paper, as the fuzzy mathematical programming approach is similar to the stochastic programming approach, those approaches are compared using the simple programming problem – portfolio selection problem. The advantages and disadvantages of the fuzzy mathematical programming approach over the stochastic programming approach are highlighted. Finally, some new approaches are briefly overviewed.

Part I: Methods and Techniques

2. Fuzzy mathematical programming

2.1. Fuzzy mathematical programming problem

Let us consider the following production planning problem (from Inuiguchi et al. [44]):

Example 1. There is a factory where two products P and Q are manufactured by two processes M and N. It takes *about* 2 *min* at Process M and *about* 6 *min* at Process N for manufacturing a batch of Product P. On the other hand, it takes *about* 3 *min* at Process M and *about* 4 *min* at Process N for manufacturing a batch of Product Q. It is desired that the working time of Process M (resp. N) is *substantially smaller than* 900 (resp. 1800) min per one term. The profit rates ($/batch) of Products P and Q are *about 7* and *about 9*, respectively. The prices ($/batch) of Products P and Q are *about* 60 and *about* 45, respectively. The factory manager requires the gross sales *substantially larger than* $22 000. Moreover, he wants to have the possibility of profit *substantially larger than* $3400. How many Products P and Q should be manufactured under such circumstances?

This problem is not clearly described as it includes uncertainty in the italic and slanted descriptions. As pointed out by some researchers (see [10,54]), two major different kinds of uncertainties, *ambiguity* and *vagueness* exist in the real life. While *ambiguity* is associated with one-to-many relations, that is, situations in which the choice between two or more alternatives is left unspecified, *vagueness* is associated with the difficulty of making sharp or precise distinctions in the world; that is, some domain of interest is vague if it cannot be delimited by sharp boundaries (see [54]).

In the above example, the slanted uncertain descriptions show the ambiguities of the true values, e.g., *about* 2 *min* shows that one value around 2 is true but not known exactly. On the other hand, the italic uncertain descriptions show the vagueness of the aspiration levels, e.g., *substantially smaller than* 900 *min* does not define a sharp boundary of a set of satisfactory values but shows that values around 900 and smaller than 900 are to some extent and completely satisfactory, respectively.

The fuzzy mathematical programming is developed for treating such uncertainties in the setting of optimization problems. The fuzzy mathematical programming can be classified into three categories in view of the kinds of uncertainties treated in the method (see [44]);

1. fuzzy mathematical programming with vagueness,
2. fuzzy mathematical programming with ambiguity,
3. fuzzy mathematical programming with vagueness and ambiguity.

The fuzzy mathematical programming in the first category was initially developed by Bellman and Zadeh [1], Tanaka et al. [103] and Zimmermann [109,110]. It treats decision making problem under fuzzy goals and constraints. The fuzzy goals and constraints represent the flexibility of the target values of objective functions and the elasticity of constraints. From this point of view, this type of fuzzy mathematical programming is called the *flexible programming*. Numerous papers were devoted to the development of this method. Many of them were overviewed by Zimmermann [111].

The second category in fuzzy mathematical programming treats ambiguous coefficients of objective functions and constraints but does not treat fuzzy goals and constraints. Dubois and Prade [8] treated systems of linear equations with ambiguous coefficients suggesting the possible application to fuzzy mathematical programming for the first time. Some years later, Tanaka et al. [97,100], Orlovski [71,72] and Ramik and Řimánek [83,84] independently proposed treatments of linear programming problems with fuzzy coefficients. Since then, many approaches to such kinds of problems have been developed. A remarkable development is done by Dubois [5]. He introduced four inequality indices between fuzzy numbers [11] based on the possibility theory [108,14] into mathematical programming problems with fuzzy coefficients. Since the fuzzy coefficients can be regarded as possibility distributions on coefficient values, this type of fuzzy mathematical programming is usually called, the *possibilistic programming*.

The last type of fuzzy mathematical programming treats ambiguous coefficients as well as vague decision maker's preference. Negoita et al. [67] were the first who formulated this type of fuzzy linear programming problem. In this model, the vague decision maker's preference is represented by a fuzzy satisfactory region and a fuzzy function value is required to be included in the given fuzzy satisfactory region. In contrast to the flexible programming, this fuzzy mathematical programming is called the *robust programming* (see [66]). Orlovski [70] formulated a general mathematical programming problem with fuzzy coefficients based on his previously proposed decision method [69] with fuzzy preference relation. Luhandjula [58,61] introduced nested target values into the objective function with fuzzy coefficients and the differences between left- and right-hand sides of the constraints with fuzzy coefficients. Inuiguchi et al. [32,29] extended the flexible programming into fuzzy coefficients case based on possibility theory. Since this type of fuzzy mathematical programming is the most generalized one, various formulations are conceivable. Inuiguchi et al. [30,36] showed that most of previous formulations including the first and the third categories are encompassed in the framework of modality constrained programming problems based on the possibility theory and the idea of chance constrained programming [64]. Similarly, Ramik et. al. [75,77–79,85,86] proposed a unified approach based on the fuzzy inequality relations.

It would take a lot of space and time to introduce all those formulations of fuzzy mathematical programming. Thus, we will restrict ourselves to describing a concise introduction to fuzzy mathematical programming using simple examples and to showing the advantages and disadvantages of the fuzzy mathematical programming approach compared with the stochastic programming approach. Through this paper, the fuzzy mathematical programming approach is investigated to reveal its properties from a practical point of view.

2.2. Fuzzy mathematical programming approach

Before describing the simple examples, let us consider the general fuzzy mathematical programming approach.

Fuzzy programming approach is illustrated in Fig. 1. As opposed to the conventional mathematical programming approach, a real world problem is first modeled using a fuzzy model (a mathematical programming model including fuzzy parameters). This fuzzy model represents an ill-posed problem, since it includes uncertain parameters. In the first phase, the fuzzy model is transformed to a usual mathematical model managing the uncertainties based on various interpretations of the problem. In the second phase, the transformed mathematical model (usual mathematical programming problem) is solved by an optimization technique. The obtained solution is optimal or efficient to the transformed mathematical model, however, it is not always reasonable (optimal or efficient) to the original fuzzy model. Thus, in the third phase, the optimality or efficiency of the solution can be examined. If the solution is improper, the fuzzy model is rebuilt to a mathematical model based on the improved interpretation and the same procedure is iterated.

The difference between the fuzzy mathematical programming approach and conventional mathematical programming approach is in the point where a fuzzy model exists between a real world optimization problem and usual mathematical model. The extra model makes Phase 3, i.e., the validity check from the fuzzy model, possible.

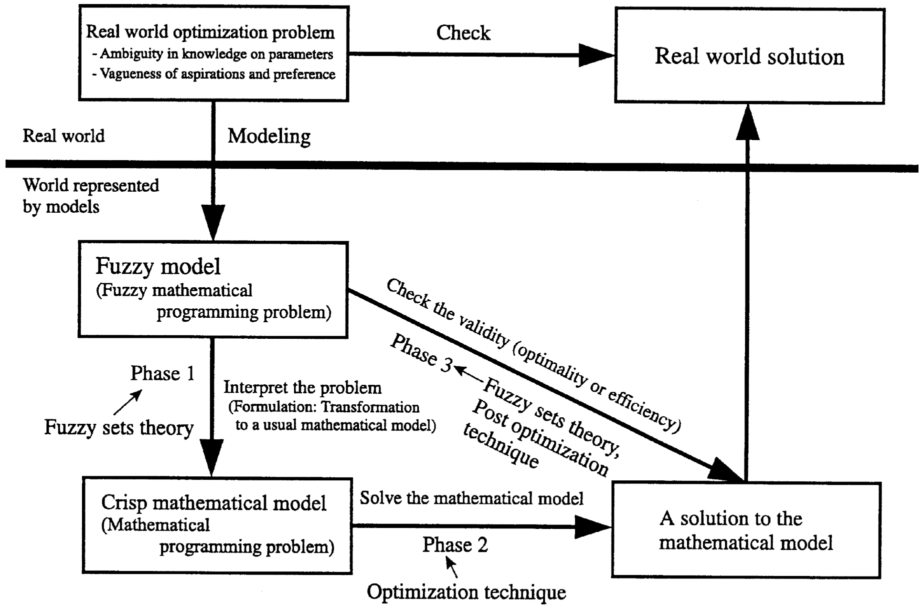

Fig. 1. Fuzzy mathematical programming approach.

3. An example and conventional mathematical programming approaches

3.1. The second example

Example 1 included ambiguous coefficients and vague aspirations. In this section, we consider a simpler problem than Example 1, which includes only ambiguous coefficients.

Example 2. In a factory, the factory manager intends to manufacture new products A and B. The total manufacturing process is composed of three processes, Processes 1, 2 and 3. The estimated processing times for manufacturing a batch of Product A at each process are the following: *about* 2 *time units* at Process 1, *about* 4 *time units* at Process 2 and *about* 1 *time unit* at Process 3. On the other hand, the processing times for manufacturing a batch of Product B at each process are as follows: *about* 3 *time units* at Process 1, *about* 2 *time units* at Process 2 and *about* 3 *time units* at Process 3. The working time at Process 1 is restricted by 240 time units, that at Process 2 is restricted by 400 time units and that at Process 3 is restricted by 210 time units. The profit rates (100$/batch) of Products A and B are *about* 5 and *about* 7, respectively. How many Products A and B should be manufactured in order to maximize the total profit?

This kind of description of the problem corresponds to 'the real world programming problem' in the fuzzy mathematical programming approach in Fig. 1.

3.2. Conventional mathematical programming approaches

Let us see what solution we get by the conventional crisp linear programming approach to Example 2.

Neglecting the ambiguity of the processing times and the profit rates, the problem of Example 2 can be formulated as

$$\begin{aligned} \text{maximize} \quad & 5x_1 + 7x_2, \\ \text{subject to} \quad & 2x_1 + 3x_2 \leqslant 240, \\ & 4x_1 + 2x_2 \leqslant 400, \\ & x_1 + 3x_2 \leqslant 210, \\ & x_1 \geqslant 0,\ x_2 \geqslant 0, \end{aligned} \tag{1}$$

where x_1 and x_2 corresponds to the amount of production of Products A and B, respectively. Solving this problem, we obtain $(x_1, x_2) = (90, 20)$. This solution reaches the upper limits of the first two constraints without violating them. However, if the true coefficients of the first two constraints take more than the estimated values, i.e., $(2, 3)$ and $(4, 2)$, this solution would violate those constraints. Thus, the solution $(x_1, x_2) = (90, 20)$ is risky in the sense of infeasibility.

Considering the ambiguity of estimated values, one may make the right-hand values more restrictive. Reducing the right-hand values to 83% of those, the problem can be formulated as

$$\begin{aligned} \text{maximize} \quad & 5x_1 + 7x_2, \\ \text{subject to} \quad & 2x_1 + 3x_2 \leqslant 199.2, \\ & 4x_1 + 2x_2 \leqslant 332, \\ & x_1 + 3x_2 \leqslant 174.3, \\ & x_1 \geqslant 0, \ x_2 \geqslant 0, \end{aligned} \tag{2}$$

where we adopt a 17% reduction because the feasible region covers almost the same size of area as that of the reduced problem obtained by the possibilistic programming approach described in the next section covers. Solving this linear programming problem, we obtain $(x_1, x_2) = (74.7, 16.6)$. Taking a ratio of x_1 to x_2 in this solution, we have $x_1/x_2 = 9/2$. This ratio is the same as that in the optimal solution to Problem (1). Generally speaking, even if we reduce the right-hand values uniformly, the ratio of x_1 to x_2 in the optimal solution does not change. It is always $x_1/x_2 = 9/2$, thus the factory manufactures Product A 4.5 times as much as Product B.

4. A possibilistic programming formulation and preliminaries

4.1. A possibilistic programming formulation

To reflect the ambiguity of estimated values in Example 2, let us express them in terms of fuzzy numbers. Interviewing the person in charge of process control, the ambiguous processing time is expressed as a fuzzy number. For example, the processing time of Product A at Process 1 described with linguistic expression 'about 2 time units', say a_1, can be restricted

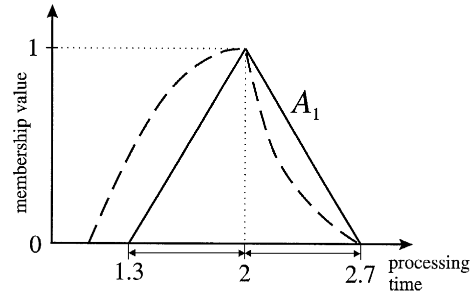

Fig. 2. A symmetric triangular fuzzy number $\langle 2, 0.7 \rangle$.

by a fuzzy number A_1 with the membership function,

$$\mu_{A_1}(r) = \max\left(0, 1 - \frac{|r-2|}{0.7}\right). \tag{3}$$

The fuzzy number A_1 is depicted in Fig. 2. As shown in Fig. 2, '2' is the most plausible value for a_1 as it takes the highest membership value. Fig. 2 also shows that a_1 is in the range $(1.3, 2.7)$ as any membership value outside this interval is zero. Moreover, the possibility of the event that a_1 is more than '2' and that of the event that a_1 is less than '2' are the same and the membership value (possibility degree) linearly decreases as the processing time departs from '2'. Thus, the fuzzy number A_1 is a symmetric triangular fuzzy number. If the person in charge of the process control section evaluates the processing time as the possibility of the event that a_1 is less than '2' is higher than the possibility of the event that a_1 is more than '2', the fuzzy number A_1 may be represented by an asymmetric fuzzy number as depicted by the broken line in Fig. 2. In this paper, for the sake of simplicity, we deal with symmetric triangular fuzzy numbers only. However, the techniques described in what follows are the same as those used in the case of asymmetric fuzzy numbers (see, for example, [94,60,33,31,89,92]. A membership function elicitation method is proposed in [45].

The symmetric triangular fuzzy number A_i in Fig. 2 can be determined by a center a_i^{c} and a spread w_{a_i}, it is represented as $A_i = \langle a_i^{\mathrm{c}}, w_{a_i} \rangle$. For example, the symmetric triangular fuzzy number A_1 in Fig. 2 is represented as $A_1 = \langle 2, 0.7 \rangle$. The membership value of the fuzzy number A_1, $\mu_{A_1}(r)$, shows the possibility degree of the event that the processing time of Product A at Process 1, a_1 is r, i.e., $a_1 = r$. In this sense, μ_{A_1} can be considered as a possibility distribution of

Table 1
Symmetric triangular fuzzy numbers

Product	A	B
Process 1	$A_1 = \langle 2, 0.7\rangle$	$B_1 = \langle 3, 0.5\rangle$
Process 2	$A_2 = \langle 4, 1.5\rangle$	$B_2 = \langle 2, 0.3\rangle$
Process 3	$A_3 = \langle 1, 0.5\rangle$	$B_3 = \langle 3, 0.3\rangle$
Profit rate	$C_1 = \langle 5, 1\rangle$	$C_2 = \langle 7, 0.7\rangle$

the processing time of Product A at Process 1 and a_1 can be regarded as a possibilistic variable restricted by the possibility distribution μ_{A_1}.

The processing time of Product A at each of the other processes, a_i, and that of Product B at each process, b_i, are also assumed to be symmetric triangular fuzzy numbers $A_i = \langle a_i^{\mathrm{c}}, w_{a_i}\rangle$ and $B_i = \langle b_i^{\mathrm{c}}, w_{b_i}\rangle$, respectively. Similarly, interviewing the person in charge of the accountants' section, the profit rate (100\$/batch) of each product, c_j is estimated as a symmetric triangular fuzzy number $C_j = \langle c_j^{\mathrm{c}}, w_{c_j}\rangle$. As a result, we obtain the symmetric triangular fuzzy numbers in Table 1. From Table 1, we can see that the spreads of the symmetric triangular fuzzy numbers of the new product A are larger than those of Product B.

The problem of Example 2 can be formulated as the following possibilistic linear programming problem;

$$\begin{aligned} \text{maximize} \quad & c_1x_1 + c_2x_2, \\ \text{subject to} \quad & a_1x_1 + b_1x_2 \leqslant 240, \\ & a_2x_1 + b_2x_2 \leqslant 400, \\ & a_3x_1 + b_3x_2 \leqslant 210, \\ & x_1 \geqslant 0, \; x_2 \geqslant 0, \end{aligned} \tag{4}$$

where a_i, b_i, $i = 1,2,3$ and c_j, $j = 1,2$, are possibilistic variables restricted by fuzzy numbers A_i, B_i, $i = 1,2,3$ and C_j, $j = 1,2$, respectively.

4.2. *Possibility distribution on a possibilistic linear function value*

Problem (4) includes linear functions of x_1 and x_2 whose coefficients are possibilistic variables. Such a function is called 'a possibilistic linear function'. Since the possibilistic variable coefficients are ambiguous parameters, the possibilistic linear function value is also ambiguous. The range of the possibilistic linear function value is restricted by a fuzzy number since the possibilistic variable coefficients are restricted by fuzzy numbers. The fuzzy number which restricts the possibilistic linear function value is defined by the extension principle (see, for example, [12]). Applying the extension principle, for example, to the objective function of Problem (4), $f_0(x_1,x_2) = c_1x_1 + c_2x_2$, the fuzzy number $F_0(x_1,x_2)$ which restricts $f_0(x_1,x_2)$ is defined by the following membership function:

$$\mu_{F_0(x_1,x_2)}(r) = \sup_{\substack{p,q \\ r = px_1 + qx_2}} \min(\mu_{C_1}(p), \mu_{C_2}(q)). \tag{5}$$

Taking into consideration the fact that C_1 and C_2 are symmetric triangular fuzzy numbers $\langle 5, 1\rangle$ and $\langle 7, 0.7\rangle$, respectively, the fuzzy number $F_0(x_1,x_2)$ also becomes a symmetric triangular fuzzy number, i.e.,

$$\begin{aligned} F_0(x_1,x_2) &= \langle 5x_1 + 7x_2,\; |x_1| + 0.7|x_2|\rangle \\ &= \langle 5x_1 + 7x_2,\; x_1 + 0.7x_2\rangle, \end{aligned} \tag{6}$$

where the second equality is from the non-negativity of x_i's of Problem (4). Generally, as is known in the literature (see, for example, [12]), if possibilistic variables y_j, $j = 1,2,\ldots,n$, are all restricted by symmetric triangular fuzzy numbers $Y_j = \langle y_j^{\mathrm{c}}, w_j\rangle$, $j = 1,2,\ldots,n$, then the fuzzy number Z which restricts $z = \sum_{j=1}^{n} k_j y_j$ is also a symmetric triangular fuzzy number,

$$Z = \left\langle \sum_{j=1}^{n} k_j y_j^{\mathrm{c}}, \sum_{j=1}^{n} |k_j| w_j \right\rangle, \tag{7}$$

where k_j, $j = 1,2,\ldots,n$, are real numbers.

Let $F_i(x_1,x_2)$ be a fuzzy number which restricts the left-hand side value of the ith constraint of (4), $f_i(x_1,x_2) = a_ix_1 + b_ix_2$. Since the fuzzy numbers A_i and B_i which restrict a_i and b_i are symmetric fuzzy numbers as shown in Table 1, $F_i(x_1,x_2)$ is also a symmetric triangular fuzzy number. Taking the non-negativity of x_i's into account, we have

$$F_1(x_1,x_2) = \langle 2x_1 + 3x_2,\; 0.7x_1 + 0.5x_2\rangle, \tag{8}$$

$$F_2(x_1,x_2) = \langle 4x_1 + 2x_2,\; 1.5x_1 + 0.3x_2\rangle, \tag{9}$$

$$F_3(x_1,x_2) = \langle x_1 + 3x_2,\; 0.5x_1 + 0.3x_2\rangle. \tag{10}$$

4.3. Indices defined by possibility and necessity measures

A possibilistic linear function value cannot be determined uniquely since its coefficients are ambiguous, i.e., non-deterministic. Thus, the objective, maximizing a possibilistic function and the constraint that a possibilistic linear function value is not greater than a certain value do not specifically make sense. To make them clear, we must introduce a specific interpretation, particularly, fuzzy inequality or ranking relations $\rho(A,B)\in[0,1]$, A and B being fuzzy sets. This belongs to Phase 1 of the fuzzy mathematical programming approach. Some well-known interpretations are reviewed and applied in the next section. In this subsection, as a basis of Phase 1, we introduce particular relations $\rho(A,B)$ called indices defined by possibility and necessity measures.

Under a possibility distribution μ_A of a possibilistic variable α, possibility and necessity measures of the event that α is in a fuzzy set B are defined as follows (see [108,14]):

$$\Pi_A(B)=\sup_r \min(\mu_A(r),\mu_B(r)), \tag{11}$$

$$N_A(B)=\inf_r \max(1-\mu_A(r),\mu_B(r)), \tag{12}$$

where μ_B is the membership function of the fuzzy set B. $\Pi_A(B)$ evaluates to what extent it is possible that the possibilistic variable α restricted by the possibility distribution μ_A is in the fuzzy set B. On the other hand, $N_A(B)$ evaluates to what extent it is certain that the possibilistic variable α restricted by the possibility distribution μ_A is in the fuzzy set B.

Let α be a possibilistic variable. In context to the above example, let $B=(-\infty,g]$, i.e., B be a crisp (nonfuzzy) set of real numbers which is not greater than g. Then we obtain the following indices by possibility and necessity measures defined by (11) and (12):

$$\begin{aligned}\mathrm{Pos}(\alpha\leqslant g)&=\Pi_A((-\infty,g])\\&=\sup\{\mu_A(r)\mid r\leqslant g\},\end{aligned} \tag{13}$$

$$\begin{aligned}\mathrm{Nes}(\alpha\leqslant g)&=N_A((-\infty,g])\\&=1-\sup\{\mu_A(r)\mid r>g\}.\end{aligned} \tag{14}$$

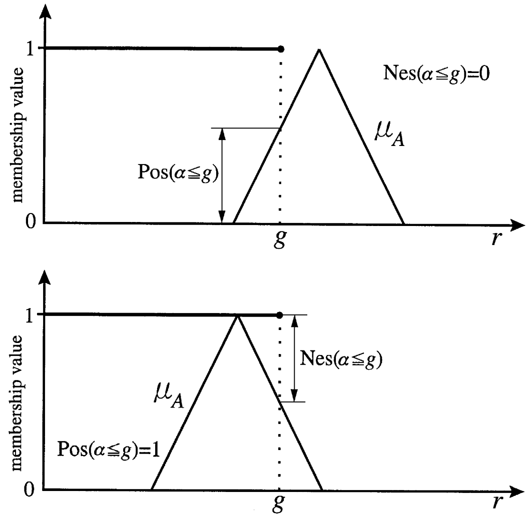

Fig. 3. Possibility and necessity degrees of $\alpha\leqslant g$.

$\mathrm{Pos}(\alpha\leqslant g)$ and $\mathrm{Nes}(\alpha\leqslant g)$ show the possibility and certainty degrees to what extent α is not greater than g. Those indices are depicted in Fig. 3.

Similarly, letting $B=[g,+\infty)$, we obtain the following two indices;

$$\begin{aligned}\mathrm{Pos}(\alpha\geqslant g)&=\Pi_A([g,+\infty))\\&=\sup\{\mu_A(r)\mid r\geqslant g\},\end{aligned} \tag{15}$$

$$\begin{aligned}\mathrm{Nes}(\alpha\geqslant g)&=N_A([g,+\infty))\\&=1-\sup\{\mu_A(r)\mid r<g\}.\end{aligned} \tag{16}$$

$\mathrm{Pos}(\alpha\geqslant g)$ and $\mathrm{Nes}(\alpha\geqslant g)$ show the possibility and certainty degrees to what extent α is not smaller than g. Those indices are depicted in Fig. 4.

Since a possibilistic linear function value $f_i(x_1,x_2)$ is a possibilistic variable restricted by $F_i(x_1,x_2)$, we can substitute $f_i(x_1,x_2)$ for α and $F_i(x_1,x_2)$ for A in (13)–(16). Thus, we can get the possibility and certainty degrees to what extent a possibilistic linear function value is not greater (smaller) than a given real number.

5. Some formulations and the solutions

As described before, the meaning of maximizing a possibilistic linear function value and the condition

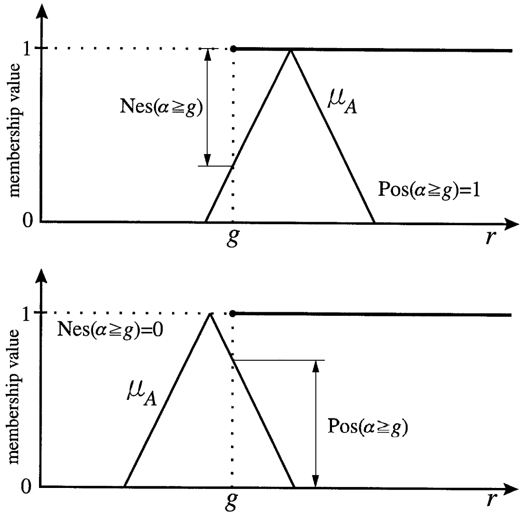

Fig. 4. Possibility and necessity degrees of $\alpha \geqslant g$.

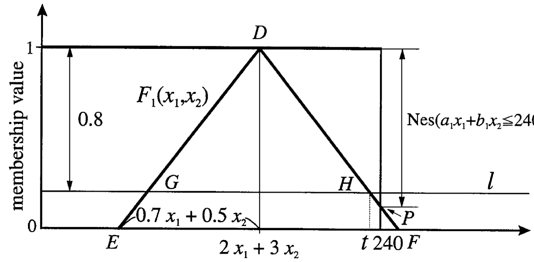

Fig. 5. $F_1(x_1,x_2)$ and $\mathrm{Nes}(a_1x_1 + b_1x_2 \leqslant 240)$.

that a possibilistic linear function value is not greater than a given fuzzy number are unclear in the traditional mathematical sense. Thus, Problem (4) is an ill-posed problem. In this section, we give specific meanings of maximizing a possibilistic linear function value and the condition that a possibilistic linear function value is not greater than a given fuzzy number, or, particularly, a crisp number, so that the ill-posed problem can be transformed to a traditional mathematical programming problem. This process is Phase 1 of the fuzzy mathematical programming approach.

Generally speaking, various interpretations are conceivable for a given fuzzy mathematical programming problem, see also e.g. [75,77–79,85,86]. Here, two well-known models are introduced. How the model can reflect the decision maker's intention is described in what follows. Before introducing the models, the treatment of the constraints, which is common to both models is described.

5.1. Treatment of the constraints

Assume that each working time cannot be extended for some reasons, e.g. for the limited workshop space part-time workers cannot be employed. In such a case, the constraints of Problem (4) should be satisfied with high certainty. If the decision maker feels that a certainty degree not less than 0.8 is high enough, the constraints of Problem (4) can be treated as follows:

$$\begin{aligned}
&\mathrm{Nes}(a_1x_1 + b_1x_2 \leqslant 240) \geqslant 0.8,\\
&\mathrm{Nes}(a_2x_1 + b_2x_2 \leqslant 400) \geqslant 0.8,\\
&\mathrm{Nes}(a_3x_1 + b_3x_2 \leqslant 210) \geqslant 0.8,\\
&x_1 \geqslant 0,\ x_2 \geqslant 0.
\end{aligned} \tag{17}$$

Let us consider the equivalent conditions to (17). To this end, we analyze the first constraint, $\mathrm{Nes}(a_1x_1 + b_1x_2 \leqslant 240) \geqslant 0.8$. From (8), the fuzzy number $F_1(x_1,x_2)$ restricting $f_1(x_1,x_2) = a_1x_1 + b_1x_2$ is a symmetric triangular fuzzy number $\langle 2x_1 + 3x_2,\ 0.7x_1 + 0.5x_2 \rangle$. This fuzzy number and the index $\mathrm{Nes}(a_1x_1 + b_1x_2 \leqslant 240)$ are depicted in Fig. 5. As shown in Fig. 5, in order to satisfy $\mathrm{Nes}(a_1x_1 + b_1x_2 \leqslant 240) \geqslant 0.8$, Point P should be under Line l. This is equivalent to the fact that t is not greater than 240. Since the isosceles triangles $\triangle DEF$ and $\triangle DGH$ are similar, we obtain

$$\begin{aligned}
t &= (2x_1 + 3x_2) + 0.8(0.7x_1 + 0.5x_2)\\
&= 2.56x_1 + 3.4x_2.
\end{aligned} \tag{18}$$

Analyzing the equivalent conditions of the other constraints of (17), we obtain the following constraints to (17):

$$\begin{aligned}
&2.56x_1 + 3.4x_2 \leqslant 240,\\
&5.2x_1 + 2.24x_2 \leqslant 400,\\
&1.4x_1 + 3.24x_2 \leqslant 210,\\
&x_1 \geqslant 0,\ x_2 \geqslant 0.
\end{aligned} \tag{19}$$

For the purpose of comparison, the feasible region of (19) and those of Problems (1) and (2) are depicted in Fig. 6. As shown in Fig. 6, the size of feasible region of Problem (2) is almost equal to that of the con-

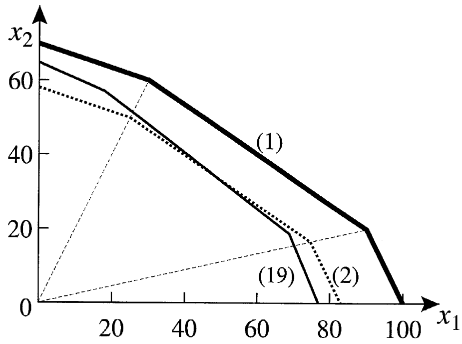

Fig. 6. Feasible regions of (1), (2) and (19).

straints (19). The constraints (19) restricts x_1 stronger, x_2 is, however, restricted weaker than the constraints of Problem (2). Particularly, to ensure feasibility, (19) more restricts the production amount of Product A which includes more ambiguous factors.

5.2. Treatment of the objectives – fractile approach

A *fractile approach* corresponds to the *Kataoka's model* [51,64] of a stochastic programming problem. Geoffrion [16] calls the Kataoka's model the *fractile criterion approach*. The fractile is defined in statistics (see, for example, [21]). By definition, p-fractile is the value u which satisfies

$$\mathrm{Prob}(X \leqslant u) = p, \tag{20}$$

where X is a random variable. In this definition, p-fractile does not generally exist for all $p \in (0,1)$. That is why we define p-fractile as the smallest value u_p of u which satisfies

$$\mathrm{Prob}(X \leqslant u) \geqslant p. \tag{21}$$

From the viewpoint of Dempster–Shafer theory of evidence [4], it is known that $\mathrm{Pos}(X \leqslant u)$ and $\mathrm{Nes}(X \leqslant u)$ can be regarded as the upper and lower bounds of $\mathrm{Prob}(X \leqslant u)$ (see [13]). In this sense, we define *p-possibility fractile* as the smallest value of u which satisfies

$$\mathrm{Pos}(X \leqslant u) \geqslant p, \tag{22}$$

and *p-necessity fractile* as the smallest value of u which satisfies

$$\mathrm{Nes}(X \leqslant u) \geqslant p. \tag{23}$$

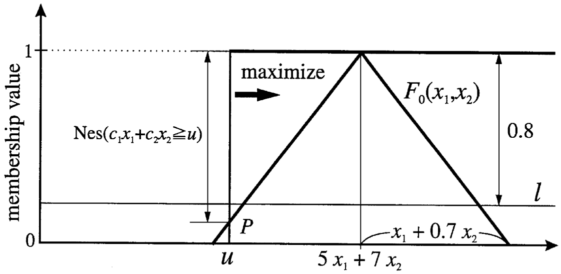

Fig. 7. The fractile optimization model.

Let us consider Example 2 again. Assume that the decision maker has a great interest in the expected profit with high certainty. Of course, the larger the expected profit is, the more preferable is the solution. If the decision maker feels the 0.8 certainty is high enough, then maximization of the objective function in Example 2 can be treated as

$$\begin{array}{ll} \text{maximize} & u \\ \text{subject to} & \mathrm{Nes}(c_1x_1 + c_2x_2 \geqslant u) \geqslant 0.8, \end{array} \tag{24}$$

which is equivalent to

$$\begin{array}{ll} \text{minimize} & v \\ \text{subject to} & \mathrm{Nes}(-c_1x_1 - c_2x_2 \leqslant v) \geqslant 0.8. \end{array} \tag{25}$$

Problem (25) is nothing but minimizing the 0.8-necessity fractile of a possibilistic variable $(-c_1x_1 - c_2x_2)$. This kind of treatment is called the *fractile approach*.

Problem (24) is illustrated in Fig. 7. As shown in Fig. 7, u is maximized under the condition that point P is under line l. By the same discussion as in Section 5.1, Problem (24) is equivalent to

$$\begin{array}{ll} \text{maximize} & u \\ \text{subject to} & 4.2x_1 + 6.44x_2 \geqslant u. \end{array} \tag{26}$$

Moreover, (26) is equivalent to

$$\text{maximize} \quad 4.2x_1 + 6.44x_2. \tag{27}$$

Finally, adding the constraints (19), Problem (4) is formulated as the following linear programming problem:

$$
\begin{aligned}
\text{maximize} \quad & 4.2x_1 + 6.44x_2 \\
\text{subject to} \quad & 2.56x_1 + 3.4x_2 \leqslant 240, \\
& 5.2x_1 + 2.24x_2 \leqslant 400, \\
& 1.4x_1 + 3.24x_2 \leqslant 210, \\
& x_1 \geqslant 0, \; x_2 \geqslant 0.
\end{aligned} \tag{28}
$$

This problem can be solved by the simplex method. The solution is obtained as $(x_1, x_2) \approx (18, 57)$.

5.3. Treatment of the objectives – modality approach

A *modality optimization model* corresponds to the *minimum-risk approach* [64] to a stochastic programming problem. The minimum-risk approach is also called the *maximum probability approach*, see [52] or, the *aspiration criterion approach* by Geoffrion [16]. A modality optimization approach is a dual approach to the fractile optimization one. Here, we assume that the decision maker puts more importance on the certainty degree comparing to the fractile approach.

For Problem (4), let us assume that the decision maker wants to maximize the certainty degree of the event that the profit is not smaller than \$45 000. This intention of the decision maker can be modeled by

$$
\text{maximize} \quad \mathrm{Nes}(c_1x_1 + c_2x_2 \geqslant 450). \tag{29}
$$

This model can be rewritten as follows with an additional variable h;

$$
\begin{aligned}
\text{maximize} \quad & h \\
\text{subject to} \quad & \mathrm{Nes}(c_1x_1 + c_2x_2 \geqslant 450) \geqslant h.
\end{aligned} \tag{30}
$$

Problem (30) is illustrated in Fig. 8. As shown in Fig. 8, h is maximized under the condition that point P is under line l. By the same discussion as in Section 5.1, Problem (30) is equivalent to

$$
\begin{aligned}
\text{maximize} \quad & h \\
\text{subject to} \quad & \frac{5x_1 + 7x_2 - 450}{x_1 + 0.7x_2} \geqslant h.
\end{aligned} \tag{31}
$$

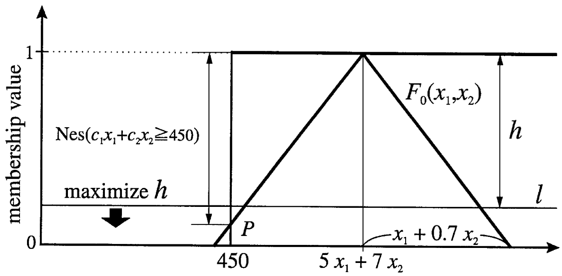

Fig. 8. The modality optimization model.

Adding the constraints (19), Problem (4) is formulated as

$$
\begin{aligned}
\text{maximize} \quad & \frac{5x_1 + 7x_2 - 450}{x_1 + 0.7x_2} \\
\text{subject to} \quad & 2.56x_1 + 3.4x_2 \leqslant 240, \\
& 5.2x_1 + 2.24x_2 \leqslant 400, \\
& 1.4x_1 + 3.24x_2 \leqslant 210, \\
& x_1 \geqslant 0, \; x_2 \geqslant 0.
\end{aligned} \tag{32}
$$

This is a linear fractional programming problem which can be transformed to a linear programming problem by the substitution

$$
t = \frac{1}{x_1 + 0.7x_2},
$$

$$
z_i = x_i t, \quad i = 1, 2,
$$

as shown by Charnes and Cooper [3]. Solving the linear programming problem,

$$
\begin{aligned}
\text{maximize} \quad & 5z_1 + 7z_2 - 450t \\
\text{subject to} \quad & 2.56z_1 + 3.4z_2 - 240t \leqslant 0, \\
& 5.2z_1 + 2.24z_2 - 400t \leqslant 0, \\
& 1.4z_1 + 3.24z_2 - 210t \leqslant 0, \\
& z_1 + 0.7z_2 = 1, \\
& z_1 \geqslant 0, \; z_2 \geqslant 0, \; t \geqslant 0,
\end{aligned} \tag{33}
$$

we obtain e.g. by simplex method the optimal solution $(z_1, z_2, t) \approx (0.311, 0.985, 0.017)$. By the reverse substitution, the optimal solution of the fractional programming problem is $(x_1, x_2) \approx (18, 57)$ which happens to be the same as that of the fractile optimization model. However, the solutions of fractile and modality optimization problems need not be always the same.

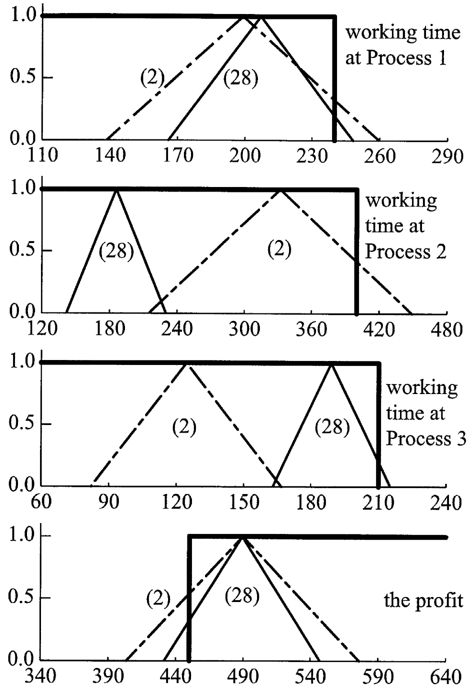

Fig. 9. Comparison between Problems (2) and (28) (or (32)).

5.4. Graphical representation of the solution and reformulation

At Phase 3 of the fuzzy mathematical programming, the obtained solution is checked whether the decision maker's intention is well-matched by the solution. In this subsection, we check the solution by its graphical representation. Moreover, reformulating Problem (4), we also describe how the fuzzy mathematical programming approach proceeds.

The possibility distributions corresponding to the solutions to Problem (2) and to Problem (28) (or (32)) are depicted in Fig. 9. From the possibility distributions with respect to the solution to Problem (2), we can observe that the certainty degree of the satisfaction of constraints on working time at Processes 1 and 2 is not high enough. Thus, we may regard the solution to Problem (2) as an ill-matched solution to the decision maker's intention.

Assume that the decision maker is not satisfied with the solution to Problem (28) (or (32)). If he/she requires that the possibility degree of the event that the profit is not smaller than \$53 000 is as high as the necessity degree of the event that the profit is not smaller than \$45 000, we can reformulate the objective function of Problem (4) as

$$\begin{aligned} \text{maximize} \quad & \min(\,\mathrm{Nes}(c_1x_1 + c_2x_2 \geqslant 450), \\ & \qquad \mathrm{Pos}(c_1x_1 + c_2x_2 \geqslant 530)). \end{aligned} \tag{34}$$

This problem can be reduced to the following problem of linear fractional programming and solved (applying the above substitution) by the simplex method:

$$\begin{aligned} \text{maximize} \quad & h \\ \text{subject to} \quad & 5z_1 + 7z_2 - 450t - h \geqslant 0, \\ & 6z_1 + 7.7z_2 - 530t - h \geqslant 0, \\ & 2.56z_1 + 3.4z_2 - 240t \leqslant 0, \\ & 5.2z_1 + 2.24z_2 - 400t \leqslant 0, \\ & 1.4z_1 + 3.24z_2 - 210t \leqslant 0, \\ & z_1 + 0.7z_2 = 1, \\ & z_1 \geqslant 0,\ z_2 \geqslant 0,\ t \geqslant 0,\ h \geqslant 0. \end{aligned} \tag{35}$$

The optimal solution after the reverse substitution is $(x_1, x_2) \approx (64.68, 21.89)$ with $h = 0.33$. This solution is depicted in Fig. 10 together with the solution to Problem (28) (or (32)). As shown in Fig. 10, compared to the solution of Problem (28), the solution of Problem (35) makes the possibility degree of the event that the profit is not smaller than \$ 53 000 a little bit higher but it makes the certainty degree of the event that the profit is not smaller than \$ 45 000 lower. The decision maker may know that he cannot offer a higher requirement than the solution to Problems (28) and (32).

Further, suppose that the decision maker wants to have a higher possibility degree of the event that the profit is not smaller than \$ 53 000 even the certainty degree of the event that the profit is not smaller than \$ 45 000 is smaller than that of the solution to Problems (29) and (33). He/She can accept a certainty degree not less than 0.5. In such a case, we can reformulate the objective function of Problem (4) as

$$\begin{aligned} \text{maximize} \quad & \mathrm{Pos}(c_1x_1 + c_2x_2 \geqslant 530)) \\ \text{subject to} \quad & \mathrm{Nes}(c_1x_1 + c_2x_2 \geqslant 450) \geqslant 0.5. \end{aligned} \tag{36}$$

By the same way, we obtain the optimal solution to Problem (36) with the constraints (19) as $(x_1, x_2) \approx (38.28, 41.76)$. This optimal solution is "between" the

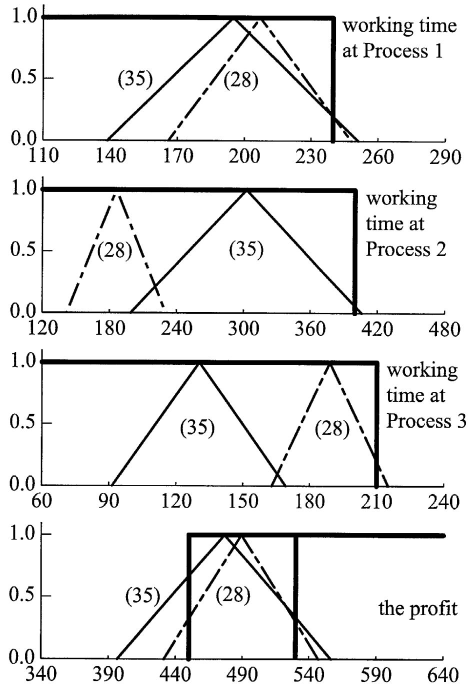

Fig. 10. Comparison between Problems (28) (or (32)) and (35).

solutions to Problems (28) (or (32)) and (35). As demonstrated above, various solutions can be obtained depending on the decision maker's intention in the fuzzy mathematical programming approach. An interactive system can be useful during the iteration process of Phases 1, 2 and 3. Through such a system, the decision maker may understand how high requirement one can ask.

Part II: Application to Portfolio Selection

6. Stochastic programming versus fuzzy mathematical programming

In the preceding sections of Part I, we have described the fuzzy mathematical programming approach through a concrete example, emphasizing that various solutions can be obtained reflecting the decision maker's intention.

Stochastic programming approaches are traditionally famous for optimization techniques under uncertainty. Someone may question the difference between fuzzy mathematical programming and stochastic programming or which is better. In this part, we compare those approaches through a portfolio selection problem in which the differences are conspicuous. Other comparison may be found e.g. in [106,95,96,91,28,38,79,86].

Generally speaking, we have the following two differences between stochastic and fuzzy mathematical programming approaches (see [28]):

1. When the random vector obeys a multivariate normal distribution, a stochastic programming problem can be solved easily. For a general distribution, a stochastic programming problem cannot usually be solved easily. On the other hand, a fuzzy mathematical programming problem can be solved easily even when the possibilistic vector is restricted by any unimodal distribution. In general, solving a fuzzy mathematical programming problem can be easier than a stochastic programming problem.
2. Suppose the uncertain variables are independent. Then only a small number of decision variables takes non-zero values in the optimal solution of the fuzzy mathematical programming problem. On the other hand, a large number of decision variables takes non-zero values in the optimal solution of the stochastic programming problem.

Now, let us look at those differences in a portfolio selection problem.

7. Portfolio selection – stochastic programming approach

7.1. Portfolio selection problem

Consider the decision problem of bond investment rate when investing a certain capital in a market where n bonds, say S_j's, are dealt with. Let c_j be the return rate of the jth bond S_j. The problem can be formulated to maximize the total return rate $\sum_{j=1}^{n} c_j x_j$ as follows:

$$\begin{aligned} &\text{maximize} && \sum_{j=1}^{n} c_j x_j \\ &\text{subject to} && \sum_{j=1}^{n} x_j = 1, \quad x_j \geqslant 0,\ j = 1,2,\ldots,n, \end{aligned} \tag{37}$$

where x_j is the decision variable which shows the investment rate to the jth bond S_j. In the real setting, one can seldom obtain the return rate without any uncertainty. Thus, the decision makers should make their decisions under uncertainty.

Such an uncertain parameter c_j has been treated as a random variable so far. Usually, c_i correlates $c_j\ (i \neq j)$, but here we assume that c_i is independent of $c_j\ (i \neq j)$ for any (i, j), $i \neq j$, in order to make the differences between stochastic and fuzzy mathematical programming approaches remarkable. Moreover, we assume that the return rate c_j obeys a normal distribution $\mathrm{N}(m_j, \sigma_j^2)$ with the mean m_j and the variance σ_j^2. Thus, the probability density function $f_{c_j}(r)$ is defined by

$$f_{c_j}(r) = \frac{1}{\sqrt{2\pi}\sigma_j} \exp\left(-\frac{(r - m_j)^2}{2\sigma_j^2} \right). \tag{38}$$

7.2. Efficiency frontier

The usual decision maker will prefer the solution which yields a large expected total return rate and a small variance. The expected total return rate corresponds to the return, while the variance corresponds to the risk. From this point of view, the portfolio selection problem can be formulated as the following bi-objective mathematical programming problem:

$$\begin{aligned}
&\text{maximize} \quad E\left(\sum_{j=1}^{n} c_j x_j\right) = \sum_{j=1}^{n} m_j x_j \\
&\text{minimize} \quad V\left(\sum_{j=1}^{n} c_j x_j\right) = \sum_{j=1}^{n} \sigma_j^2 x_j^2 \\
&\text{subject to} \quad \sum_{j=1}^{n} x_j = 1, \quad x_j \geqslant 0,\ j = 1, 2, \ldots, n.
\end{aligned} \tag{39}$$

Usually, we cannot obtain a complete optimal solution which optimizes both objective functions, i.e. the expected total return rate and the variance, simultaneously in Problem (39). Thus, a Pareto optimal solution, such that there is no feasible solution which makes both objective function values better at the same time, is calculated. Generally there exist a lot of Pareto optimal solutions to Problem (39).

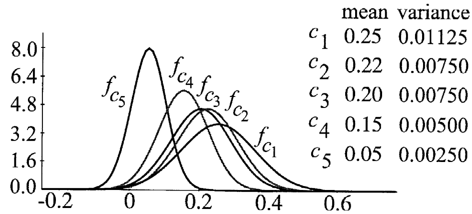

Fig. 11. Normal distributions.

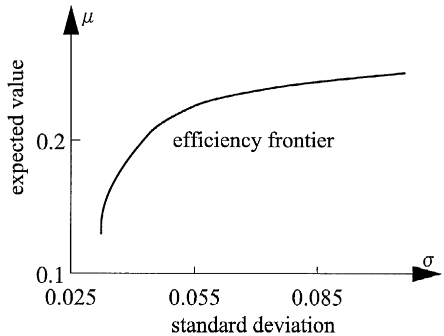

Fig. 12. Efficiency frontier.

For example, the set of Pareto optimal solutions are obtained as shown in Fig. 12 to Problem (39) with 5 bonds whose return rates obey normal distributions indicated in Fig. 11. Strictly speaking, in Fig. 12, the expected values and the standard deviations of Pareto optimal solutions are plotted, where a standard deviation is the square root of a variance. Such a curve is called an *efficiency frontier*. A large expected value and a small standard deviation are preferable. The left and upper region of the efficiency frontier is the infeasible region. Thus, the efficiency frontier is the border obtained by improving the expected value and the standard deviation (variance) in the feasible region.

7.3. Markowitz model

The original model of the portfolio selection problem was proposed by Markowitz [62]. The model is the so called *V-model* [64] in stochastic programming. To obtain a Pareto optimal solution to Problem (39),

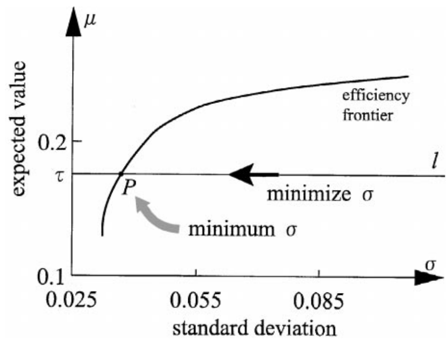

Fig. 13. Markowitz model.

he treated the problem so as to minimize the variance keeping the expected value at a given constant τ, i.e.,

$$
\begin{aligned}
&\text{minimize} \quad V\left(\sum_{j=1}^{n} c_j x_j\right) = \sum_{j=1}^{n} \sigma_j^2 x_j^2 \\
&\text{subject to} \quad E\left(\sum_{j=1}^{n} c_j x_j\right) = \sum_{j=1}^{n} m_j x_j = \tau \qquad (40)\\
&\qquad \sum_{j=1}^{n} x_j = 1, \quad x_j \geqslant 0,\ j = 1, 2, \ldots, n.
\end{aligned}
$$

This problem is a quadratic programming problem. Thus it can be solved easily (see, for example, [15]).

This model can be explained by using Fig. 13. Namely, this model finds the solution corresponding to point P which minimizes the standard deviation along line l on which the expected value is constantly τ. Applying this model with $\tau = 0.18$ to the portfolio selection problem with normal distributions depicted in Fig. 11, the optimal solution is obtained as $(x_1, x_2, x_3, x_4, x_5) \approx (0.1767, 0.2325, 0.2109, 0.2350, 0.1449)$. A distributive investment solution is obtained so as to avert the risk. The defect of this model is that a solution indicating an improperly large investment in an inefficient bond with a small variance may be obtained on condition that we select too small τ. This follows from the fact that a small variance does not imply a large expected value. Indeed, the obtained solution with respect to $\tau = 0.18$ indicates a 14.49% investment in the fifth bond which may be regarded as inferior.

7.4. Kataoka's model

We may apply the Kataoka's model to Problem (37) with random return rates. In this model, we maximize z such that the probability of the event that the total return rate is not smaller than z is at least $1 - \alpha$, i.e.,

$$
\begin{aligned}
&\text{maximize} \quad z \\
&\text{subject to} \quad \text{Prob}\left(\sum_{j=1}^{n} c_j x_j \geqslant z\right) \geqslant 1 - \alpha, \qquad (41)\\
&\qquad \sum_{j=1}^{n} x_j = 1, \quad x_j \geqslant 0,\ j = 1, 2, \ldots, n.
\end{aligned}
$$

Since we assume that each return rate obeys a normal distribution, Problem (41) can be reduced to the following mathematical programming problem (see, for example, [64]):

$$
\begin{aligned}
&\text{maximize} \quad \sum_{j=1}^{n} m_j x_j + k_\alpha \sqrt{\sum_{j=1}^{n} \sigma_j^2 x_j^2} \\
&\text{subject to} \quad \sum_{j=1}^{n} x_j = 1, \quad x_j \geqslant 0,\ j = 1, 2, \ldots, n,
\end{aligned} \qquad (42)
$$

where k_α is the α-fractile of the standard normal distribution N(0, 1), i.e., we have $\Pr(X \leqslant k_\alpha) = \alpha$ ($X \sim$ N(0, 1)). Compared to Problem (40), solving Problem (42) is much more difficult. However, it is known that Problem (42) can be solved by a repetitional use of quadratic programming when $\alpha < 0.5$ (see, for example, [64]).

Problem (42) can be explained by Fig. 14. Namely, the y-intercept z is maximized under the constraint that the linear function $\mu = -k_\alpha \sigma + z$ intersects the efficiency frontier. Thus, we obtain a solution corresponding to point Q. Applying this model with $\alpha = 0.05$ to the portfolio selection problem with normal distributions depicted in Fig. 11, we have $(x_1, x_2, x_3, x_4, x_5) \approx (0.3103, 0.3429, 0.2613, 0.0855, 0)$. Even though we set $\alpha = 0.05$, a small number, we obtaine $x_5 = 0$. From this, we can be convinced that the fifth bond is inferior.

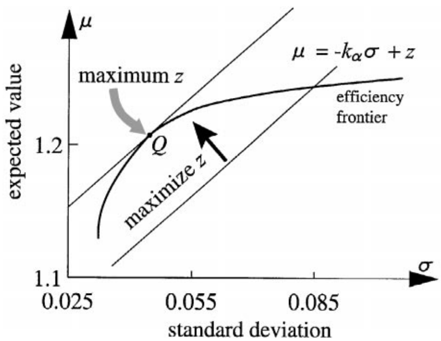

Fig. 14. Kataoka's model.

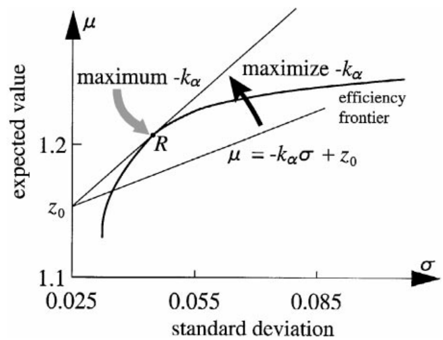

Fig. 15. Minimum-risk model.

7.5. *Minimum-risk model*

We apply the minimum-risk model to Problem (37) with random return rates. In contrast to Kataoka's model, we maximize the probability of the event that the total return rate is not smaller than a predetermined value z_0 in this model, i.e.,

$$\begin{aligned} &\text{maximize} \quad \mathrm{Prob}\left(\sum_{j=1}^{n} c_j x_j \geqslant z_0\right) \\ &\text{subject to} \quad \sum_{j=1}^{n} x_j = 1, \quad x_j \geqslant 0, \ j = 1, 2, \ldots, n. \end{aligned} \tag{43}$$

Since we assume that each return rate obeys a normal distribution, Problem (43) can be reduced to the following mathematical programming problem (see, for example, [64]):

$$\begin{aligned} &\text{maximize} \quad \frac{\sum_{j=1}^{n} m_j x_j - z_0}{\sqrt{\sum_{j=1}^{n} \sigma_j^2 x_j^2}}, \\ &\text{subject to} \quad \sum_{j=1}^{n} x_j = 1, \quad x_j \geqslant 0, \ j = 1, 2, \ldots, n. \end{aligned} \tag{44}$$

This problem can be solved by a repetitional use of quadratic programming when there exists a feasible solution such that $\sum_{j=1}^{n} m_j x_j > z_0$ (see, for example, [64]).

Problem (44) can be explained by Fig. 15. Namely, the slope $-k_\alpha$ is maximized under the constraint that the linear function $\mu = -k_\alpha \sigma + z_0$ intersects the efficiency frontier. Thus, we obtain a solution corresponding to point R. Applying this model with $z_0 = 0.18$ to the portfolio selection problem with normal distributions depicted in Fig. 11, we have $(x_1, x_2, x_3, x_4, x_5) \approx (0.4380, 0.3750, 0.1870, 0, 0)$.

8. Portfolio selection – possibilistic programming approach

8.1. *Bi-objective programming problem*

In the previous section, we assume that each return rate c_j is a random variable. In this section, we assume that each return rate c_j is a possibilistic variable. Corresponding to the normal distributions in Fig. 11, we have normal fuzzy numbers C_j with the membership functions defined by

$$\mu_{C_j}(r) = \exp\left(-\frac{(r - c_j^{\mathrm{c}})^2}{w_j^2}\right), \tag{45}$$

where c_j^{c} is a center value of the normal fuzzy number C_j and takes the same values as the mean m_j of the corresponding normal distribution. On the other hand, w_j is a spread of the normal fuzzy number and is equal to $\sqrt{2}\sigma_j$, where σ_j is a standard deviation of the corresponding normal distribution. The normal fuzzy numbers corresponding to the normal distributions in Fig. 11 are depicted in Fig. 16.

From now on, we assume that c_j's in Problem (37) are mutually independent possibilistic variables restricted by normal fuzzy numbers C_j's.

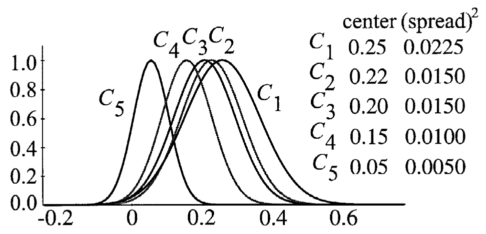

Fig. 16. Normal fuzzy numbers.

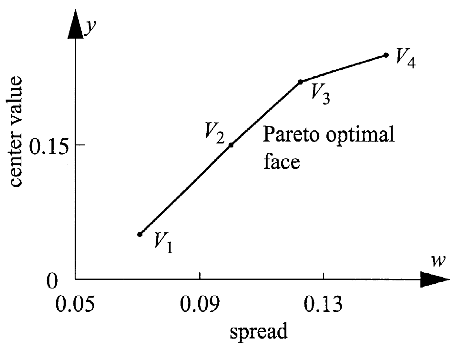

Fig. 17. Pareto optimal face.

We apply a fuzzy mathematical programming approach to the portfolio selection problem in what follows.

Since the center values and spreads correspond to the means and variances (standard deviations), respectively, the following bi-objective programming problem is conceivable in analogy to Problem (39):

$$\begin{aligned} &\text{maximize} \quad \sum_{j=1}^{n} c_j^{\mathrm{c}} x_j \\ &\text{minimize} \quad \sum_{j=1}^{n} w_j x_j \\ &\text{subject to} \quad \sum_{j=1}^{n} x_j = 1, \quad x_j \geqslant 0, \quad j = 1, 2, \ldots, n. \end{aligned} \tag{46}$$

Problem (46) preserves the linearity of the original problem (37) while Problem (39) does not preserve it, i.e., Problem (39) is quadratic.

Pareto optimal solutions to Problem (46) with the normal fuzzy numbers of Fig. 16 are obtained as shown in Fig. 17. In Fig. 17, we can see that the Pareto optimal solution set forms a polygonal line. The vertices V_1, V_2, V_3 and V_4 correspond to concentrate investments in bonds S_5, S_4, S_2 and S_1, respectively.

8.2. Spread minimization model

In analogy to Problem (40), we may have

$$\begin{aligned} &\text{minimize} \quad \sum_{j=1}^{n} w_j x_j \\ &\text{subject to} \quad \sum_{j=1}^{n} c_j^{\mathrm{c}} x_j = \tau, \\ &\qquad\qquad \sum_{j=1}^{n} x_j = 1, \quad x_j \geqslant 0, \quad j = 1, 2, \ldots, n. \end{aligned} \tag{47}$$

This problem is called the *spread minimization model.*

Whereas Problem (40) is a quadratic programming problem, Problem (47) is a linear programming one. Thus, Problem (47) can be solved easier than Problem (40). Problem (47) has only two constraints other than non-negativity constraints on the decision variables. From the fundamental theorems of linear programming (see, for example, [19]), usually only two decision variables are positive at the optimal solution to Problem (47) even if n is large. This means that Problem (47) suggests an investment only in two bonds, i.e., a semi-concentrated investment.

For example, applying this model with $\tau = 0.18$ to a portfolio selection with the normal fuzzy numbers of Fig. 16, we obtain $(x_1, x_2, x_3, x_4, x_5) \approx (0, 0.4286, 0, 0.5714, 0)$. This solution shows an investment in bonds S_2 and S_4. In another way, using Fig. 18, we can understand that the solution corresponding to Point P on the line segment from Vertex V_2 to Vertex V_3 is optimal. Since Vertices V_2 and V_3 correspond to the bonds S_4 and S_2, the solution means an investment in bonds S_4 and S_2. As demonstrated above, the solution of this model does not suggest a distributive investment.

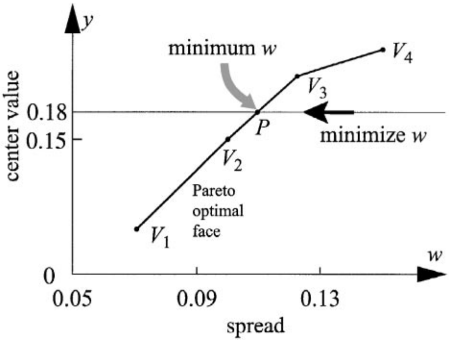

Fig. 18. Spread minimization model.

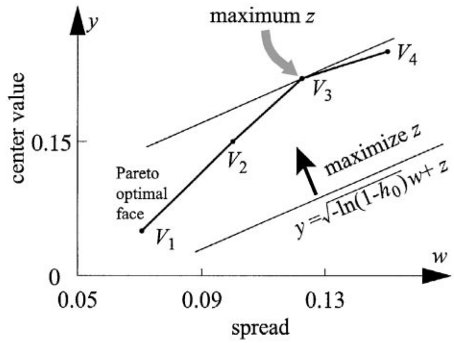

Fig. 19. Fractile optimization model.

8.3. Fractile approach

Applying the fractile approach to Problem (37) with normal fuzzy number coefficients, we have

$$\begin{aligned}&\text{maximize} \quad z\\&\text{subject to} \quad \mathrm{Nes}\left(\sum_{j=1}^{n} c_j x_j \geqslant z\right) \geqslant h_0,\\&\qquad \sum_{j=1}^{n} x_j = 1, \quad x_j \geqslant 0, \ j=1,2,\ldots,n,\end{aligned} \tag{48}$$

where $h_0 \in (0,1]$ is a predetermined value. This problem can be reduced to the following linear programming problem:

$$\begin{aligned}&\text{maximize} \quad \sum_{j=1}^{n} c_j^{\mathrm{c}} x_j - \sqrt{-\ln(1-h_0)} \sum_{j=1}^{n} w_j x_j\\&\text{subject to} \quad \sum_{j=1}^{n} x_j = 1, \quad x_j \geqslant 0, \ j=1,2,\ldots,n.\end{aligned} \tag{49}$$

This linear programming problem corresponds to the Kataoka's model in stochastic programming approach. Whereas the Kataoka's model is reduced to a non-linear programming problem (42), the fractile optimization model is reduced to a linear programming problem. Consequently, the fractile optimization model yields a simpler reduced problem than the Kataoka's model.

Problem (49) has only one constraint besides the non-negativity constraints on the decision variables. From the fundamental theorem of linear programming, usually, only one decision variable takes a positive value at the optimal solution to Problem (49). Thus, the solution suggests an investment only in a bond S_j which has the largest objective function coefficient $(c_j^{\mathrm{c}} - \sqrt{-\ln(1-h_0)}w_j)$. For example, applying this model with $h_0 = 0.9$ to a portfolio selection problem with the normal fuzzy numbers of Fig. 16, we obtain $(x_1,x_2,x_3,x_4,x_5) \approx (0,1,0,0,0)$. Namely, the solution suggests an investment only in the bond S_2. Fig. 19 shows how the solution is obtained. At vertex V_3, the y-intercept z of a line $y = \sqrt{-\ln(1-h_0)}w + z$ is maximal. Vertex V_3 corresponds to the bond S_2.

Therefore, by the fractile approach of fuzzy mathematical programming, a risky concentrated investment solution is obtained.

8.4. Modality approach

Applying the Modality approach to Problem (37) with the normal fuzzy numbers, we have

$$\begin{aligned}&\text{maximize} \quad \mathrm{Nes}\left(\sum_{j=1}^{n} c_j x_j \geqslant z_0\right)\\&\text{subject to} \quad \sum_{j=1}^{n} x_j = 1, \quad x_j \geqslant 0, \ j=1,2,\ldots,n,\end{aligned} \tag{50}$$

where $z_0 \in (0,1]$ is a predetermined value. This problem can be reduced to the following linear fractional

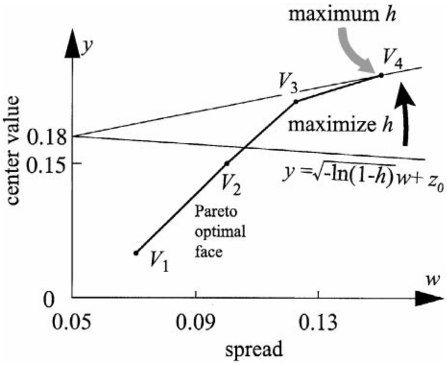

Fig. 20. Modality optimization model.

programming problem:

$$\begin{aligned} &\text{maximize} \quad \frac{\sum_{j=1}^{n} c_j^{\mathrm{c}} x_j - z_0}{\sum_{j=1}^{n} w_j x_j}, \\ &\text{subject to} \quad \sum_{j=1}^{n} x_j = 1, \quad x_j \geqslant 0, \ j = 1, 2, \ldots, n. \end{aligned} \tag{51}$$

Again, this linear fractional programming problem can be transformed to a linear programming problem since the denominator of the objective function is positive for any feasible solution. Indeed, defining $t = 1/\sum_{j=1}^{n} w_j x_j$ and $y_j = t x_j$ as shown in Section 5.3, we can reduce Problem (51) to the following linear programming problem:

$$\begin{aligned} &\text{maximize} \quad \sum_{j=1}^{n} c_j^{\mathrm{c}} y_j - z_0 t \\ &\text{subject to} \quad \sum_{j=1}^{n} w_j y_j = 1, \\ &\qquad \sum_{j=1}^{n} y_j = t, \\ &\qquad t \geqslant 0, \ y_j \geqslant 0, \ j = 1, 2, \ldots, n. \end{aligned} \tag{52}$$

This model corresponds to the minimum-risk model in stochastic programming. Whereas the minimum-risk model is solved by a repetitional use of quadratic programming, the modality optimization model is solved by linear programming.

Problem (51) also yields a concentrated investment solution. This can be explained in Fig. 20 by using an example with normal fuzzy numbers of Fig. 16. We assume $z_0 = 0.18$. As shown in Fig. 20, Problem (51) is equivalent to the problem of maximizing h under the condition that a line $y = \sqrt{-\ln(1-h)}w + z_0$ intersects the Pareto optimal face. Since maximizing h is equivalent to maximizing the slope $\sqrt{-\ln(1-h)}$, the maximum is attained at a vertex. In the current model, the maximum h is obtained at vertex V_4 which corresponds to a concentrated investment in the bond S_1.

Analogously to the fractile approach, by the modality optimization model of fuzzy mathematical programming approach, a risky concentrated investment solution is obtained.

9. Possibilistic programming treats more uncertain parameters

Inuiguchi and Sakawa [38] showed that a possibilistic linear programming problem with a quadratic membership function is equivalent to a stochastic programming problem with a multivariate normal distribution. In this section, we describe that a possibilistic linear programming problem with independent possibilistic variables is equivalent to a stochastic linear programming problem with unknown correlation coefficients between normal random variables.

Let ρ_{ij} be the correlation coefficient between normal random variables c_i and c_j. The covariance matrix Σ can be represented by

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & \cdots & \rho_{2n}\sigma_2\sigma_n \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1n}\sigma_1\sigma_n & \rho_{2n}\sigma_2\sigma_n & \cdots & \sigma_n^2 \end{pmatrix}. \tag{53}$$

$\sum_{j=1}^{n} c_j x_j$ obeys a multivariate normal distribution given by

$$N\left(\sum_{j=1}^{n} m_j x_j, \ \boldsymbol{x}^{\mathrm{t}} \Sigma \boldsymbol{x}\right), \tag{54}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{t}}$ is a column vector.

In the current problem, we assume that ρ_{ij}'s are unknown elements from the interval $[-1, 1]$ (see, for example, [21]). In decision-making under uncertainty, taking care of the worst case, it is important to avert the risk. Reflecting this consideration, we may chose the most uncertain probability distribution of $\sum_{j=1}^{n} c_j x_j$ among all possible probability distributions obtained by changing every ρ_{ij} in $[-1,1]$. Because $\boldsymbol{x}$ is non-negative, the probability distribution with respect to $\rho_{ij}=1,\ i<j$, is selected as the most uncertain one. Particularly, we have the following equality for any $\boldsymbol{x}$:

$$\max_{\rho_{ij}\in[-1,1],\, i<j} \boldsymbol{x}^{\mathrm{t}} \Sigma \boldsymbol{x} = \left(\sum_{j=1}^{n} \sigma_j x_j \right)^2. \tag{55}$$

Let us consider a portfolio selection problem with unknown correlation coefficients between normal random return rates. Taking care of the worst case, we may have the following problem corresponding to Problem (39):

$$\begin{aligned} &\text{maximize} \quad E\left(\sum_{j=1}^{n} c_j x_j \right) = \sum_{j=1}^{n} m_j x_j \\ &\text{minimize} \quad \max_{\rho_{ij}\in[-1,1],\ i<j} V\left(\sum_{j=1}^{n} c_j x_j \right) \\ &\qquad\qquad = \left(\sum_{j=1}^{n} \sigma_j x_j \right)^2 \\ &\text{subject to} \quad \sum_{j=1}^{n} x_j = 1, \quad x_j \geqslant 0,\ j=1,2,\ldots,n. \end{aligned} \tag{56}$$

Problem (56) is equivalent to

$$\begin{aligned} &\text{maximize} \quad \sum_{j=1}^{n} m_j x_j \\ &\text{minimize} \quad \sum_{j=1}^{n} \sqrt{2}\sigma_j x_j \\ &\text{subject to} \quad \sum_{j=1}^{n} x_j = 1, \quad x_j \geqslant 0,\ j=1,2,\ldots,n. \end{aligned} \tag{57}$$

From the correspondences between m_j and c_j^{c} and that between $\sqrt{2}\sigma_j$ and w_j, Problem (57) is nothing but Problem (46).

In the same way, applying Markowitz model, Kataoka's model and the minimum-risk model to the portfolio selection problem with unknown correlation coefficients between normal random return rates, we obtain the equivalent problems to Problems (47), (49) and (51) under the equation $k_\alpha = \sqrt{-2\ \ln(1-h)}$. This is true even in a general linear programming problem under uncertainty. Therefore, we can regard a possibilistic linear programming problem with independent possibilistic variables as a stochastic linear programming problem with unknown correlation coefficients between normal random variables.

As shown above, when we treat a stochastic programming problem with unknown correlation coefficients, we cannot always obtain a distributive investment solution. This means that a distributive investment solution is not obtained from the decision procedures of Markowitz, Kataoka's and the minimum-risk problems, but from a property of the probability measure (the definition of probabilistic independence). In the next section, we introduce a decision procedure from which we can obtain a distributive investment solution to a portfolio selection problem with normal fuzzy numbers (see [40]).

10. Minimax regret model

Now, we discuss why a distributive investment solution under independent return rate assumption is preferred by a decision maker who has an uncertainty (risk) averse attitude. We can observe at least the following two reasons:

(a) *Property of a measure.* Assume that we have two bonds and the return rate of each bond obeys the same marginal distribution. Consider the event that the total return rate is not less than a certain value. When the measure of the event under a distributive investment solution is greater than that under a concentrated investment solution, the distributive investment solution should be preferable.

(b) *The worst regret criterion.* Suppose that the decision maker has invested his money in a bond according to a concentrated investment solution. If the return rate of another bond becomes better than that of the invested bond, as a result, the decision maker may feel a regret. At the decision making stage, we cannot know the return rate determined in the future. Thus, any concentrated investment solution may bring a regret to the decision maker. In this sense, if the decision maker is interested in minimizing the worst regret which may be undertaken, a distributive investment solution must be preferable.

As described in the preceding section, in stochastic programming approaches, a distributive investment solution is obtained. This is because of (a), i.e., the Property of a measure. Indeed, we have

$$\mathrm{Prob}(\lambda X_1 + (1-\lambda)X_2 \geqslant k) > \mathrm{Prob}(X_i \geqslant k), \quad \forall \lambda \in (0,1),\ i=1,2, \tag{58}$$

when random variables X_1 and X_2 obey the same marginal normal (probability) distribution, the correlation coefficient ρ_{12} is less than 1 and k is a constant larger than the expected value. In fuzzy programming approaches, we could not obtain a distributive investment solution since possibility and necessity measures do not have the property mentioned in (a). For possibility and necessity measures, we have

$$\begin{aligned} &\mathrm{Pos}(\lambda X_1 + (1-\lambda)X_2 \geqslant k) \\ &\quad = \mathrm{Pos}(X_i \geqslant k),\ \forall \lambda \in [0,1],\ i=1,2, \\ &\mathrm{Nes}(\lambda X_1 + (1-\lambda)X_2 \geqslant k) \\ &\quad = \mathrm{Nes}(X_i \geqslant k),\ \forall \lambda \in [0,1],\ i=1,2, \end{aligned} \tag{59}$$

where X_1 and X_2 are mutually independent possibilistic variables restricted by the same marginal possibility distribution.

Now let us introduce the worst regret criterion into a portfolio selection problem with normal fuzzy numbers so that we can obtain a distributive investment solution.

Suppose that a decision maker is informed about the determined return rates $\boldsymbol{c}$ after he has invested his money in bonds according to a feasible solution $\boldsymbol{x}$ to Problem (37), he will have a regret $r(\boldsymbol{x};\boldsymbol{c})$ which can be quantified as

$$r(\boldsymbol{x};\boldsymbol{c}) = \max\{\boldsymbol{c}^{\mathrm{t}}\boldsymbol{y} - \boldsymbol{c}^{\mathrm{t}}\boldsymbol{x} \mid \boldsymbol{e}^{\mathrm{t}}\boldsymbol{y} = 1,\ \boldsymbol{y} \geqslant 0\}. \tag{60}$$

Regret $r(\boldsymbol{x};\boldsymbol{c})$ is the difference between the optimal total return rate with respect to $\boldsymbol{c}$ and the obtained total return rate $\boldsymbol{c}^{\mathrm{t}}\boldsymbol{x}$.

At the decision-making stage, the decision maker cannot know the return rate $\boldsymbol{c}$ determined in the future but a possibility distribution $\mu_C(\boldsymbol{c})$ is supposed to be known. By the extension principle [12], a possibility distribution $\mu_{R(\boldsymbol{x})}$ on regrets can be defined as

$$\mu_{R(\boldsymbol{x})}(r) = \sup\ \{\mu_C(\boldsymbol{c}) \mid r = r(\boldsymbol{x};\boldsymbol{c}),\ \boldsymbol{c} = (c_1, c_2, \ldots, c_n)^{\mathrm{t}}\}. \tag{61}$$

We regard the portfolio selection problem with fuzzy numbers as a problem of minimizing a regret $R(\boldsymbol{x})$ with a possibility distribution $\mu_{R(\boldsymbol{x})}$, i.e.,

$$\begin{aligned} &\text{minimize} \quad R(\boldsymbol{x}) \\ &\text{subject to} \quad \sum_{j=1}^{n} x_j = 1, \quad x_j \geqslant 0,\ j=1,2,\ldots,n. \end{aligned} \tag{62}$$

Since $R(\boldsymbol{x})$ is a possibilistic variable restricted by a possibility distribution $\mu_{R(\boldsymbol{x})}$, (62) is also a possibilistic programming problem. We apply the fractile model to Problem (62) so that, given h_0, Problem (62) is formulated as

$$\begin{aligned} &\text{minimize} \quad z \\ &\text{subject to} \quad N_{R(\boldsymbol{x})}(\{r \mid r \leqslant z\}) \geqslant h_0, \\ &\qquad \sum_{j=1}^{n} x_j = 1, \quad x_j \geqslant 0,\ j=1,2,\ldots,n. \end{aligned} \tag{63}$$

Inuiguchi and Sakawa [40] showed that Problem (63) is reduced to a linear programming problem. In the case of normal fuzzy numbers, we have

$$\begin{aligned} &\text{minimize} \quad q \\ &\text{subject to} \\ &\sqrt{-\ln(1-h_0)}\left(w_i x_i - \sum_{\substack{j=1 \\ i \neq j}}^{n} w_j x_j\right) + \sum_{j=1}^{n} c_j^{\mathrm{c}} x_j + q \\ &\quad \geqslant c_i^{\mathrm{c}} + \sqrt{-\ln(1-h_0)}\, w_i, \quad i=1,2,\ldots,n, \\ &\sum_{j=1}^{n} x_j = 1, \quad x_j \geqslant 0,\ j=1,2,\ldots,n. \end{aligned} \tag{64}$$

Applying this model with $h_0 = 0.8$ to the portfolio selection problem with the normal fuzzy numbers of Fig. 16, we obtain $(x_1, x_2, x_3, x_4, x_5) \approx (0.4080, 0.3067, 0.2528, 0.0325, 0)$. Although the solution does not suggest an investment in the bond S_5, it is a distributive investment solution on the bonds S_1 to S_4.

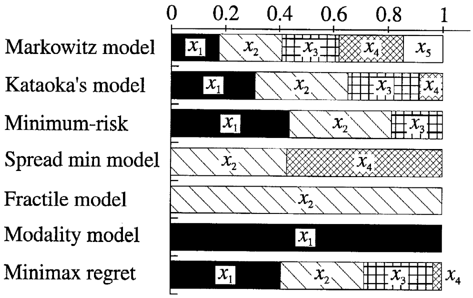

Fig. 21. A band graph of the solutions.

The worst regret criterion can be introduced into a stochastic programming problem, however, it is difficult to solve the reduced problem. On the other hand, in a fuzzy mathematical programming approach, introduction of a new criterion is usually easier.

The solutions are compared by a band graph in Fig. 21. The length of a rectangle with a variable name, x_j, shows the portion that the solution suggests to invest in the corresponding bond S_j. As shown in Fig. 21, the minimax regret solution takes a middle position between Kataoka's and minimum-risk models. Considering the effort to calculate the solution as well as the convincibility of the solution, the minimax regret model would be the best among the seven models.

11. New trends in fuzzy mathematical programming

As described in the previous sections, the fuzzy mathematical programming approaches have some advantages in the tractability of the reduced problem over the stochastic programming approaches. The basic developments of fuzzy mathematical programming is almost done but there is still many topics to be investigated. Some of the new trends in fuzzy mathematical programming are briefly reviewed in what follows. As can be seen from the literature, the fuzzy mathematical programming is still being developed widely and deeply.

11.1. Optimality and efficiency

An objective function with fuzzy coefficients has been treated based on a goal attainment criterion or a ranking criterion between fuzzy numbers in many papers, so far. There were not so many proposals to treat the fuzzy objective function based on the optimality concept. Luhandjula [59] first treated a fuzzy vector objective function based on the optimality (more exactly, efficiency) concept. He proved several theorems. [37,42] extended the optimality for a single objective function and the efficiency for multiple objective functions based on the possibility theory. They also presented optimality and efficiency tests for a given feasible solution.

Following Inuiguchi and Sakawa [37,42], two kinds of optimality can be defined based on the possibility theory: the possibly and necessarily optimal solutions. Briefly speaking, the possibly optimal solution is a solution which is optimal for at least one possible objective coefficient vector. On the other hand, the necessarily optimal solution is a solution which is optimal for all possible objective coefficient vectors. A necessarily optimal solution does not always exist but it seems to be the most reasonable solution. When no necessarily optimal solution exists, there exist a lot of possibly optimal solutions. In this case, we should provide a selection method to chose from the possibly optimal solutions.

The possible and necessary optimality tests will be useful at Phase 3 of the fuzzy mathematical programming approach. Some extension and other optimal and efficiency concepts are also conceivable (see, for example, [35,34,27]).

11.2. Minimax regret model

In the preceding section, we described the minimax regret model. This model has some interesting property, i.e. a minimax regret solution is possibly optimal and it is also necessarily optimal when a necessarily optimal solution exists. From this good property, a minimax regret criterion can be applied to linear programming problems with interval and fuzzy objective coefficients and solution methods based on a relaxation procedure have been proposed (see [39,41]).

Another model which has the same property, called the achievement rate approach, has also been developed (see [43]). Moreover, this model can be used for fuzzy linear programming problems with recourse (see [49] for stochastic programming problems with recourse).

11.3. Interactions among possibilistic variables

So far, most of fuzzy mathematical programming techniques have been developed for non-interactive (independent) fuzzy numbers. Recently, the researchers have been trying to introduce the interaction between fuzzy numbers. Four possible avenues to treat the interaction were proposed.

Quadratic membership function: Originally, this expression of a multivariate possibility distribution was proposed for fuzzy linear regression methods by Celmiš [2], later on, it was developed by Tanaka and Ishibuchi [101,102]. A quadratic membership function is the one obtained by normalizing the mode of a multivariate normal distribution. A quadratic membership function $\mu_C(\boldsymbol{c})$ is defined by

$$\mu_C(\boldsymbol{c}) = L((\boldsymbol{c}-\boldsymbol{d})^{\mathrm{t}} U^{-1}(\boldsymbol{c}-\boldsymbol{d})), \tag{65}$$

where $\boldsymbol{d} = (d_1, \dots, d_n)^{\mathrm{t}}$ is the most conceivable vector for $\boldsymbol{c}$, U is an $n \times n$ symmetrical positive-definite matrix representing the interactions among objective coefficients. U^{-1} is the inverse matrix of U, $L : \boldsymbol{R} \to [0,1]$ is a reference function which is a non-increasing function in the domain $[0, +\infty)$ such that $L(r) = L(-r)$ and $L(0) = 1$.

An introduction of the above approach to fuzzy linear programming problems was done by Inuiguchi [28] and Inuiguchi and Sakawa [38]. They showed the equivalence between fuzzy linear programming and stochastic linear programming with a multivariate normal distribution. Ida [27] has also investigated possible and necessary optimality tests for a fuzzy linear programming problem with quadratic membership functions. Tanaka and Guo [99] applied quadratic membership functions to portfolio selection problems.

Since a quadratic membership function has similar properties to a multivariate normal distribution, it can be easily introduced to many kinds of problem. However, it cannot express any interaction between possibilistic variables with high proximity.

T-norm based extension: In the independent possibilistic variables case, a joint (multivariate) possibility distribution can be obtained by taking the minimum among the marginal possibility distributions. In the t-norm-based extension, we replace the minimum operation with a t-norm (see, for example, [10]). Thus, a joint (multivariate) possibility distribution μ_C can be expressed by

$$\mu_C(\boldsymbol{c}) = T(\mu_{C_1}(c_1), T(\mu_{C_2}(c_2), T(\dots, \mu_{C_n}(c_n))\dots), \tag{66}$$

where μ_{C_i} is a marginal possibility distribution of the ith possibilistic variable.

The possibilistic variables restricted by the joint possibility distribution are not totally (possibilistic) independent but, in certain sense, independent. For example, if the t-norm is a product, the possibility variables are quite similar to independent random variables. Moreover, a multivariate possibility distribution cannot be always expressed by marginal possibility distributions. From such a viewpoint, possibilistic variables are said to be weakly independent (non-interactive) if the joint possibility distribution is decomposable to marginal possibility distributions by a t-norm (see [10]). Rommelfanger et al. [90,88] proposed the use of Yager's parameterized t-norm for solving fuzzy linear programming problems. By using of Yager's parameterized t-norm, the decision maker can obtain a more flexible instrument for expressing his/her risk mentality.

Canonical fuzzy number: A canonical fuzzy number is defined formally by Nakamura [65] and Ramík and Nakamura [81,82]. A canonical fuzzy number C can be defined by the following membership function μ_C:

$$\mu_C(\boldsymbol{c}) = \min_{i=1,2,\dots,n} L\left(\frac{\sum_{j=1}^{n} d_{ij}c_j - d_{i0}}{\alpha_i}\right), \tag{67}$$

where L is a reference function, d_{ij} is a real number and α_i is a positive real number.

When $d_{ii} = 1$, $i = 1, 2, \dots, n$, and $d_{ij} = 0$, $i \neq j$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, n$, a membership function of a canonical fuzzy number is reduced to a possibility distribution of independent possibilistic variables. Thus, a membership function of a canonical fuzzy number can be considered as an extension of a possibility distribution of independent possibilistic variables but it can treat interactions among possibilistic variables. Since the linearity is not lost in a canonical fuzzy number, it is useful for modeling fuzzy linear programming problems.

Scenario decomposition: Scenario decomposition is originally introduced in stochastic programming (see, for example, [49]). Ohta et al. [68] introduced this method into a fuzzy linear programming problem in order to treat interactions among uncertain variables.

In this method, we first select a scenario variable, s, which influences to some uncertain variables. Then, the possible realizations of the scenario variable s is determined. To each possible realization of s, we assign a possible range of each uncertain variable. Under the realization of s, we assume that uncertain variables are independent.

A multivariate distribution can be expressed by an inference model such as 'if s is $\sim$ then the range of uncertain variable is $\sim$'. For each value of s, we can treat a fuzzy mathematical programming problem in the traditional way. That is why this model is useful to treat interactive possibilistic variables.

Ohta et al. [68] and Katagiri and Ishii [50] treated a scenario variable as a random variable. We can treat it as a possibilistic variable, too.

11.4. *Fuzzy combinatorial programming*

The treatments of fuzzy mathematical programming problems have been already well-developed. Such developments were done mostly in fuzzy linear programming problems. Thus, many researchers are trying to extend the application area to fuzzy combinatorial programming problems.

Nowadays, meta-heuristic methods, such as genetic algorithms [63], simulated annealing [55], tabu search [17,18] and so on, are popular. Some researches are applying meta-heuristic methods to fuzzy combinatorial programming problems, such as fuzzy scheduling problems [46,47,104,23], fuzzy project selection problems [93,53] and so forth. Using a meta-heuristic method, one can obtain only approximate solutions even to a complex problem.

On the other hand, other researchers are developing a theoretical approach to fuzzy combinatorial programming problems (see [48,22,24–26]). Dubois et al. [6] treated such a combinatorial problem in the framework of flexible constraint satisfaction problem [7].

11.5. *Fuzzy solutions*

A fuzzy mathematical programming problem includes the ambiguity of coefficients and/or the vagueness of aspirations. In such an uncertain environment, one may think how much we can make the uncertain solution reflect the uncertainty of the problem setting. Such an uncertain solution is called the *fuzzy solution*.

Fuzzy solutions are initially investigated by Verdegay [105] and Tanaka and Asai [98]. Verdegay treated a mathematical programming problem under fuzzy constraints. An element of the fuzzy solution with membership degree h is the solution which optimizes the objective function under h-level set of fuzzy constraints. On the other hand, Tanaka and Asai treated a system of inequalities with fuzzy coefficients. They calculated a fuzzy solution with the widest spread such that the solution satisfy the system of inequalities to a given degree.

Real world problems are not usually so easily formulated as mathematical models or fuzzy models. Sometimes qualitative constraints and/or objectives are almost impossible to represent in mathematical forms. In such a situation, a fuzzy solution satisfying the given mathematically represented requirements are very useful in a sense of weak focus in the feasible area. The decision maker can select the final solution from the fuzzy solution considering implicit and mathematically weak requirements.

The possibility and necessarily optimal (or efficient) solution sets can be considered as fuzzy solutions to a fuzzy mathematical programming problem with fuzzy coefficients.

The fuzzy solutions have not yet been investigated considerably. Recently, some researchers [80,20] started to tackle the fuzzy solution problem. Several advanced methods may emerge in the near future.

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