Quantum optics exercises *

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Abstract

These exercises have been done in the course of Quantum Optics that is studied in the 4th course of Physics in the Universitat Autònoma de Bellatera.

Contents

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1 Classical theory of the light-matter interaction

1.1 Maxwell's equations in vacuum

Exercise 1 *Using the definitions of the vector potential* $\vec{A}$ $(\vec{B} = \vec{\nabla} \times \vec{A})$ *and the scalar potential* ϕ $\left(\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\phi\right)$*, find the wave equation for the vector potential* $\vec{A}$*:*

Solution 1 We need the Maxwell's equations in vacuum:

$$\nabla \cdot \vec{B} = 0 \tag{1}$$

$$\nabla \cdot \vec{E} = 0 \tag{2}$$

$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{3}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{4}$$

We assume that the wave is moving in vacuum, so we can impose:

$$\nabla \cdot \vec{A} = \phi = 0 \tag{5}$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} \tag{6}$$

Using the definition of the vector potential and substitute equation 6 into equation 3, we obtain the equation

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}$$

Now we have to use the relation: $\vec{\nabla} \times \left(\vec{\nabla} \times \quad\right) = -\nabla^2 \quad + \nabla(\nabla \quad)$, and use the equation 5 and we finally find:

$$-\nabla^2 \vec{A} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} \qquad \rightarrow \qquad \nabla^2 \vec{A} + \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = 0$$

We have defined the velocity of the light in vacuum, c, like: $c = \sqrt{\frac{1}{\mu_0 \epsilon_0}}$

Exercise 2 *Verify, by substitution, that* $\vec{E}(\vec{r}, t) = \vec{E_0} f(\vec{k}\vec{r} - \omega t)$ *is a solution of the wave equation* $\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E_0}}{\partial t^2}$

Solution 2 We know that $\vec{E_0} \in \Re$ and $\left|\vec{k}\right| = \frac{\omega}{c}$. Substituting it into the wave equation:

$$\nabla^2 \vec{E} = \nabla \left(\vec{E_0} \cdot \vec{k} f(\vec{k}\vec{r} - \omega t)\right) = \vec{E_0} \cdot |\vec{k}|^2 f(\vec{k}\vec{r} - \omega t)$$

$$\frac{\partial^2 \vec{E_0}}{\partial t^2} = \frac{\partial}{\partial t} \left(\vec{E_0} \cdot (-\omega) f(\vec{k}\vec{r} - \omega t)\right) = \vec{E_0} \cdot \omega^2 f(\vec{k}\vec{r} - \omega t)$$

If we composite the two results, we find:

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}_0}{\partial t^2} = 0$$
$$\vec{E}_0 \cdot |\vec{k}|^2 f(\vec{k}\vec{r} - \omega t) - \frac{1}{c^2}\vec{E}_0 \cdot \omega^2 f(\vec{k}\vec{r} - \omega t) = 0$$
$$k^2 - \frac{\omega^2}{c^2} = 0 \quad \rightarrow \quad k = \frac{\omega}{c}$$

Exercise 3 *We have to verify that for $\vec{E}_1(\vec{r},t) = \vec{E}_{01} f_1(\vec{k}_1\vec{r} - \omega_1 t)$ and $\vec{E}_2(\vec{r},t) = \vec{E}_{02} f_2(\vec{k}_2\vec{r} - \omega_2 t)$ being solutions of Maxwell's equations, then $\vec{E}_3 = \vec{E}_1 + \vec{E}_2$ will be a solution too*

Solution 3 We have to verify that if $\vec{E}_1$ and $\vec{E}_2$ are two solutions of the Maxwell's equations then $\vec{E}_3 = \vec{E}_1 + \vec{E}_2$ is a solution too.

We can define $\vec{B}_1$ and $\vec{B}_2$ as the magnetic fields associated with the electric fields $\vec{E}_1$ and $\vec{E}_2$, respectively. Hence, the two solutions observe the Maxwell's equations:

$\vec{\nabla} \cdot \vec{B}_1 = 0$	$\vec{\nabla} \cdot \vec{B}_2 = 0$
$\vec{\nabla} \cdot \vec{E}_1 = 0$	$\vec{\nabla} \cdot \vec{E}_2 = 0$
$\vec{\nabla} \times \vec{E}_1 = -\frac{\partial \vec{B}_1}{\partial t}$	$\vec{\nabla} \times \vec{E}_2 = -\frac{\partial \vec{B}_2}{\partial t}$
$\vec{\nabla} \times \vec{B}_1 = \mu_0\epsilon_0 \frac{\partial \vec{E}_1}{\partial t}$	$\vec{\nabla} \times \vec{B}_2 = \mu_0\epsilon_0 \frac{\partial \vec{E}_2}{\partial t}$

Now, we want to confirm if $\vec{E}_3$ and $\vec{B}_3$ observes the Maxwell's equations:

$$\underline{\vec{\nabla} \cdot \vec{E}_3} = \vec{\nabla} \cdot \left(\alpha\vec{E}_1 + \beta\vec{E}_2\right) = \alpha\left(\vec{\nabla} \cdot \vec{E}_1\right) + \beta\left(\vec{\nabla} \cdot \vec{E}_2\right) = 0 + 0 = \underline{0}$$

$$\underline{\vec{\nabla} \cdot \vec{B}_3} = \vec{\nabla} \cdot \left(\alpha\vec{B}_1 + \beta\vec{b}_2\right) = \alpha\left(\vec{\nabla} \cdot \vec{B}_1\right) + \beta\left(\vec{\nabla} \cdot \vec{B}_2\right) = 0 + 0 = \underline{0}$$

$$\underline{\vec{\nabla} \times \vec{E}_3} = \vec{\nabla} \times \left(\alpha\vec{E}_1 + \beta\vec{E}_2\right) = \alpha\left(\vec{\nabla} \times \vec{E}_1\right) + \beta\left(\vec{\nabla} \times \vec{E}_2\right) =$$
$$\alpha\left(-\frac{\partial \vec{B}_1}{\partial t}\right) + \beta\left(-\frac{\partial \vec{B}_2}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\alpha\vec{B}_1 + \beta\vec{B}_2\right) = \underline{-\frac{\partial \vec{B}_3}{\partial t}}$$

$$\underline{\vec{\nabla} \times \vec{B}_3} = \vec{\nabla} \times \left(\alpha\vec{B}_1 + \beta\vec{B}_2\right) = \alpha\left(\vec{\nabla} \times \vec{B}_1\right) + \beta\left(\vec{\nabla} \times \vec{B}_2\right) =$$
$$\alpha\mu_0\epsilon_0\left(-\frac{\partial \vec{B}_1}{\partial t}\right) + \beta\mu_0\epsilon_0\left(-\frac{\partial \vec{B}_2}{\partial t}\right) = -\mu_0\epsilon_0\frac{\partial}{\partial t}\left(\alpha\vec{B}_1 + \beta\vec{B}_2\right) = \underline{-\mu_0\epsilon_0\frac{\partial \vec{B}_3}{\partial t}}$$

Exercise 4 *Find*

$$\frac{\partial E_0}{\partial z} + \frac{1}{c}\frac{\partial E_0}{\partial t} = 0$$
$$\frac{\partial \phi}{\partial z} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0$$

form the wave equation, applying the SVAP approximation:

$$\left|\frac{\partial E_0}{\partial t}\right| << \omega E_0 \qquad \left|\frac{\partial \omega}{\partial t}\right| << \omega$$
$$\left|\frac{\partial E_0}{\partial z}\right| << k E_0 \qquad \left|\frac{\partial \omega}{\partial z}\right| << k$$

Solution 4 The one-dimensional wave equation is stated by: $E'' - \frac{1}{c^2}\ddot{E} = 0$, we suppose the solution $E(x,t) = E_0(x,t)e^{i(kx-\omega t-\phi(x,t))}$ and we use the relations: $E' = \frac{\partial E}{\partial x}$; $E'' = \frac{\partial^2 E}{\partial x^2}$; $\dot{E} = \frac{\partial E}{\partial t}$; $\ddot{E} = \frac{\partial^2 E}{\partial t^2}$. Substituting the supposed solution in the wave's equation:

$$\begin{aligned} E' &= E_0' e^{i(\quad)} + E_0 e^{i(\quad)}(k-\phi') \\ &\Downarrow \\ E'' &= E_0'' e^{i(\quad)} + E_0' e^{i(\quad)}(k-\phi') + E_0' e^{i(\quad)}(k-\phi') + \\ &\quad E_0 e^{i(\quad)}(k-\phi')^2 + E_0 e^{i(\quad)}(-\phi'') \end{aligned}$$

If $E_0' << kE_0$ and $\phi' << k$ then it follows that $E_0'' \approx 0$, $\phi'' \approx 0$ and $E_0'\phi' \approx 0$. Applying these approximations, we obtain:

$$E'' = 2kE_0' e^{i(\quad)} + k^2 E_0 e^{i(\quad)} - 2k\phi' E_0 e^{i(\quad)}$$

Now we calculate the second term of the wave's equation:

$$\begin{aligned} \dot{E} &= \dot{E}_0 e^{i(\quad)} + E_0 e^{i(\quad)}\left(-\omega - \dot{\phi}\right) \\ &\Downarrow \\ \ddot{E} &= \ddot{E}_0 e^{i(\quad)} + \dot{E}_0 e^{i(\quad)}\left(-\omega - \dot{\phi}\right) + \dot{E}_0 e^{i(\quad)}\left(-\omega - \dot{\phi}\right) + \\ &\quad E_0 e^{i(\quad)}\left(-\omega - \dot{\phi}\right)^2 + E_0 e^{i(\quad)}\left(-\ddot{\phi}\right) \end{aligned}$$

From the SVAP approximation: $\dot{E}_0 << \omega E_0$ and $\dot{\phi} << \omega$, follows that $\ddot{E}_0 \approx 0$, $\ddot{\phi} \approx 0$ and $\dot{E}_0\dot{\phi} \approx 0$. Applying these approximations, we obtain:

$$\ddot{E} = -2\omega \dot{E}_0 e^{i(\quad)} + \omega^2 E_0 e^{i(\quad)} + 2\omega\dot{\phi} E_0 e^{i(\quad)}$$

Substituting E'' and $\ddot{E}$ in the wave's equation, we obtain:

$$\begin{gathered} 2kE_0' e^{i(\quad)} + k^2 E_0 e^{i(\quad)} - 2k\phi' E_0 e^{i(\quad)} - \\ \frac{1}{c^2}\left(-2\omega \dot{E}_0 e^{i(\quad)} + \omega^2 E_0 e^{i(\quad)} + 2\omega\dot{\phi} E_0 e^{i(\quad)} = 0\right) \end{gathered}$$

Using the relation between k, ω and c: $k = \frac{\omega}{c}$, we can eliminate the second term of E'' and $\ddot{E}$; we can eliminate the term $e^{i(\quad)}$ too:

$$\begin{gathered} 2kE_0' - 2k\phi' E_0 - \frac{1}{c^2}\left(-2\omega\dot{E}_0 + 2\omega\dot{\phi}E_0\right) = 0 \\ 2kE_0' - 2k\phi' E_0 - \frac{1}{c^{\not 2}}\left(-2\not c k\dot{E}_0 + 2\not c k\dot{\phi}E_0\right) = 0 \\ E_0' + \frac{1}{c}\left(\dot{E}_0\right) - E_0\left(\phi' + \frac{1}{c}\left[\dot{\phi}\right)\right] = 0 \end{gathered}$$

$E_0(x,t)$ and $\phi(x,t)$ are two independent functions of x and t, so, finally we find the equations:

$$E_0' + \frac{1}{c}\left(\dot{E}_0\right) = 0 \qquad \phi' + \frac{1}{c}\left(\dot{\phi}\right) = 0$$

1.2 Maxwell's equations in a material medium

Exercise 5 *Deduce the wave's equation for the electric field, when you use the Maxwell's equations for a material medium*

Solution 5 The Maxwell's equations for a material medium are:

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{1}$$

$$\vec{\nabla} \cdot \vec{D} = \sigma_{free} \tag{2}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3}$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_{free} + \frac{\partial \vec{D}}{\partial t} \tag{4}$$

where $\vec{D} = \epsilon \vec{E} + \vec{P}$ and $\vec{H} = \frac{\vec{B}}{\mu} - \vec{M}$

At first, we calculate the rotational of equation 3:

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

Now we can use the relation: $\vec{\nabla} \times \left(\vec{\nabla} \times \quad\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \quad\right) - \nabla^2$ and the commutation between the rotational operator and partial time derivative: $\left[\vec{\nabla}\times, \frac{\partial}{\partial t}\right]$ $= 0$

$$\vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{B}\right) \tag{5}$$

Now, we use the equations 4 and the definition of $\vec{H}$: $\vec{H} = \frac{\vec{B}}{\mu} - \vec{M}$, to obtain an expression of $\vec{\nabla} \times \vec{B}$:

$$\vec{\nabla} \times \vec{H} = \vec{J}_{free} + \frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla} \times \left(\frac{\vec{B}}{\mu} - \vec{M}\right) = \vec{J}_{free} + \frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu \vec{M} + \mu \vec{J}_{free} + \mu \frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu \vec{M} + \mu \vec{J}_{free} + \mu \frac{\partial \vec{P}}{\partial t} + \mu\epsilon \frac{\partial \vec{E}}{\partial t}$$

Substituting in equation 5:

$$\vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\mu \vec{M} + \mu \vec{J}_{free} + \mu \frac{\partial \vec{P}}{\partial t} + \mu\epsilon \frac{\partial \vec{E}}{\partial t}\right)$$

We can simplify this expression, assuming a non-magnetic material where $\vec{M}=0$, this circumstance implies $\mu\left(\vec{\nabla}\times\vec{M}\right)=0$. We can assume $\vec{\nabla}\cdot\vec{E}=0$ if we think that the vector $\vec{E}$ doesn't change more in the direction of propagation of the wave. Finally, we find wave's equation for the electric field in a material medium:

$$\nabla^2\vec{E}-\frac{1}{v^2}\frac{\partial^2\vec{E}}{\partial t^2}=\mu\left(\frac{\partial\vec{J}_{free}}{\partial t}+\frac{\partial^2\vec{P}}{\partial t^2}\right)$$

where $v=\frac{1}{\sqrt{\epsilon\mu}}$ is the propagation velocity

Exercise 6 *Find the equations:*

$$E_0'+\frac{1}{c}\dot{E}_0=\frac{-k}{2\epsilon}Im\left\{\mathrm{P}\right\}=\frac{k}{2\epsilon}N(z)dV \tag{6}$$

$$E_0\left(\phi'+\frac{1}{c}\dot{\phi}\right)=\frac{-k}{2\epsilon}Re\left\{\mathrm{P}\right\}=\frac{-k}{2\epsilon}N(z)dU \tag{7}$$

from the wave equation in a material medium and applying the SVAP approximation.

Solution 6 At first, we consider the absence of free current in the material, so we can eliminate the term $\mu\frac{\partial\vec{J}_{free}}{\partial t}=0$ from the wave equation found in the last exercise:

$$\nabla^2\vec{E}-\frac{1}{v^2}\frac{\partial^2\vec{E}}{\partial t^2}=\mu\left(\frac{\partial^2\vec{P}}{\partial t^2}\right)$$

Now, we consider a solution of the wave's equation:

$$\vec{E}(z,t)=\frac{1}{2}\vec{e}_xE_0(z,t)e^{k\cdot z-\omega t-\phi(z,t)}$$

1.3 Lorentz's classical model of the light-matter interaction

Exercise 7 *Prove that* $x(t)=x_0e^{\frac{-\gamma}{2}\pm\sqrt{\frac{\gamma^2}{4}-\omega_0^2}t}=x_0e^{-\frac{\gamma}{2}\pm i\omega_0t}$ *is a solution of the differential equation* $\ddot{x}+\gamma\dot{x}+\omega_0^2x=0$

Solution 7

Exercise 8 *Verify that*

$$x(t)=\frac{e}{m}E_0\left[\frac{\omega_0^2-\omega^2}{\left(\omega_0^2-\omega^2\right)^2+\left(\gamma\omega\right)^2}cos(\omega t)+\frac{\gamma\omega}{\left(\omega_0^2-\omega^2\right)^2+\left(\gamma\omega\right)^2}sin(\omega t)\right]$$

is a solution of the equation: $\ddot{x}+\gamma\dot{x}+\omega_0^2x=\frac{e}{m}E_0cos(\omega t)$

Solution 8

Exercise 9 *Explain why sky is blue, using the equations of the diffusion. Help: the colour of the sky is a phenomena of diffusion produced in the troposphere (25-80 km) by the presence of the ozone molecule with resonance frequency in the ultraviolate*

Solution 9

1.4 The susceptibility

Exercise 10 *Check ...:*

1. *the FWHM (Full width at half medium) of* $Re\{\alpha\}$ *is* γ
2. *and the maximum value of the refraction index is produced a half height ob the lorentzian curve*

Solution 10

1.5 Propagation of non-monochromatic's waves

1.6 Introduction to non-lineal optics

Exercise 11 *Deduce the magnitude order of the non-linear susceptibility of second order. Help: Take the model of the hydrogen atom and consider that the non-linear terms have a similar value than the lineal contribution of the electric field when it is similar than the atomic electric field*

Solution 11

Exercise 12 *In practice, the typical electric field of tie between electrons and ions is* $E \approx 10^{10} V m^{-1}$. *Which intensity do we need to observe non-linear effects? Do these intense light source exist? Do we need the intense light source to observe non-linear effects, for example, the second harmonic generation?*

Solution 12

Exercise 13 *Prove the expression* $n = n_0 + n_2 I$, *determining the relation between* n_2 *and* $\chi^{(3)}$

Solution 13

1.7 Susceptibility non-lineal of a classical an harmonic oscillator

2 Semi classical theory of the light-matter interaction

2.1 Termical radiation and Planck's hypothesis

Exercise 14 *Which thermodynamic arguments can we use to affirm that* $\rho(\nu, T)$ *is a universal function? Help: Clausius formulate the second law of the thermodynamics in the form: "The transformations which final result is to pass heat from a body to other body more hot are impossible"*

Solution 14 We can realize a mental experiment. We have to imagine two separated cavities showed in the figure 1. These cavities are surrounded by a thermal bath with a fixed temperature T, the two thermal bath have the same temperature.

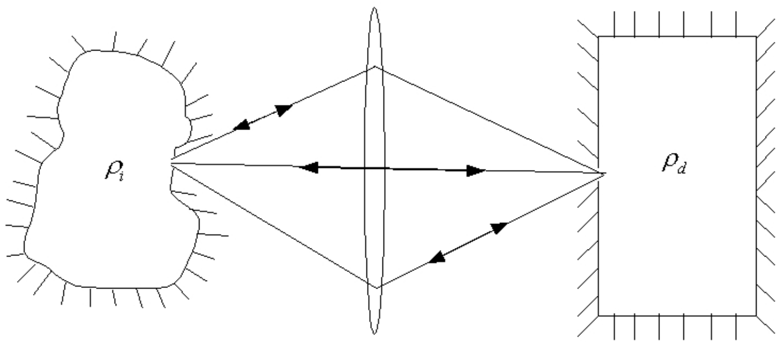

Figure 1: Mental experiment

We suppose that the radiation density of each cavity are in equilibrium with the cavity, so the number of absorption and emission are the assume. So we, have a radiation density which is constant for each frequency.

Now, we make a little hole in each cavity, so the radiation can go out the cavity.

Our experiment consist to place one cavity in front of the other cavity. With a sofisticated optical process, we can bring the radiation which go out from ρ_i and bring it to the cavity ρ_d. At the same time, we are bringing the radiation which go out from ρ_d and bringing it to the cavity ρ_i.

Using the second law of the termodynamics (the Clausius formulation), we can see that the two cavities are in thermal equilibrium because the two cavities have the same temperature

Exercise 15 *Prove that the wave's equation for the electric field can be rewritten when we use* $\vec{E}(x, y, z, t) = \vec{u}(x, y, z)A(t)$

Solution 15

Exercise 16 *Prove that the expressions:*

$$u_x = d_x \; cosk_x x \; sink_y y \; sink_z z$$
$$u_y = d_y \; sink_x x \; cosk_y y \; sink_z z$$
$$u_z = d_z \; sink_x x \; sink_y y \; cosk_z z$$

are the solutions of the Helmholtz's equation: $\nabla^2 \vec{u} = -k^2 \vec{u}$

Solution 16

Exercise 17 *Find the condition:* $\vec{d} \cdot \vec{k} = 0$*, where* $\vec{d} = (d_x, d_y, d_z)$*, from the equation:* $\vec{\nabla} \cdot \vec{E} = 0$

Solution 17

Exercise 18 *Demonstrate (using two lines):* $\langle E \rangle = \frac{h\nu}{exp[h\nu/k_B T]-1}$

Solution 18 We have the operation for calculate $\langle E \rangle$

$$\langle E \rangle = \frac{\sum_{n=0}^{\infty} nh\nu e^{-nh\nu/k_B T}}{\sum_{n=0}^{\infty} e^{-nh\nu/k_B T}}$$

So we have to calculate two sums, one in the numerator and another in the denominator. Beginning with the sum of the denominator:

$$\sum_{n=0}^{\infty} e^{-nh\nu/k_B T} = 1 + e^{-h\nu/k_B T} + e^{-2h\nu/k_B T} + ...$$

We can see that is a progressive geometric sum that it's solved in a lot of books [1], so:

$$\sum_{n=0}^{\infty} e^{-nh\nu/k_B T} = \frac{1 - e^{-nh\nu/k_B T}}{1 - e^{-h\nu/k_B T}} = \frac{1}{1 - e^{-h\nu/k_B T}}$$

The numerator of the last equation is equal to 1 because $n \to \infty$ and $e^{-\infty} = 0$

For the sum of the numerator, we have to work more than before. We must see that it isn't a progressive geometric sum, so we have to transform the sum in a progressive geometric sum. We can see the following relation:

$$\sum_n n e^{-nx} = \frac{\partial}{\partial x}\left(\sum_n e^{-nx}\right) = \frac{\partial}{\partial x}\left(\frac{1}{1 - e^{-x}}\right) = \frac{x e^{-x}}{(1 - e^{-x})^2}$$

Identifing $x = h\nu/k_B T$, we can solve the sum of the numerator, and find the valor of $\langle E \rangle$:

$$\begin{aligned}\langle E \rangle &= h\nu \frac{\sum_{n=0}^{\infty} n e^{-nh\nu/k_B T}}{\sum_{n=0}^{\infty} e^{-nh\nu/k_B T}} \\ &= h\nu \frac{\frac{e^{-h\nu/k_B T}}{\left(1 - e^{-h\nu/k_B T}\right)^2}}{\frac{1}{1 - e^{-h\nu/k_B T}}} \\ &= h\nu \frac{e^{-h\nu/k_B T}}{\left(1 - e^{-h\nu/k_B T}\right)} \\ &= h\nu \frac{1}{\left(e^{h\nu/k_B T}\right) - 1}\end{aligned}$$

Exercise 19 *How many photons per way, for optics frequency and ambiental temperature, is there?*

Solution 19 If we want to find the number of photons per way, we must use the expression of Bose-Einstein statistics, so the photons are bosons:

$$\langle n \rangle = \frac{1}{e^{\frac{h\nu}{k_B T}} - 1} = \frac{1}{e^{\frac{hc}{\lambda k_B T}} - 1}$$

[1]Spiegel, Liu, Abellanas - Frmulas y tablas de matemtica aplicada

For ambiental temperature, we can suppose $T = 300K$ and for optical frequency we can use $\lambda = 500nm$. Substituting in the last expression, we find:

$$\langle n \rangle = \left(exp \left[\frac{6.62 \cdot 10^{-34} Js \quad 3 \cdot 10^8 m/s}{500 \cdot 10^{-9} m \quad 1.38 \cdot 10^{-23} J/K \quad 300K} \right] - 1 \right)^{-1}$$

$$\langle n \rangle = (exp\,[95] - 1)^{-1} \approx e^{-95} << 1$$

Exercise 20 *Demonstrate that $B = c\rho/4\pi$ where B is the brightness ($B = \frac{I}{\pi}$), and ρ is the energy density: $\rho = \frac{1}{2}\epsilon E^2(t) + \frac{1}{2}\mu H^2(t)$*

Solution 20

Exercise 21 *Deduce the Wien law ($\lambda_{max} T = const$) from the expression:*

$$B_\lambda d\lambda = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda k_B T} - 1} d\lambda$$

Solution 21 Observing the plot of the brightness, we can see that the plot has a maximum valor for one wavelength, to find this valor we must derive the expression of the brightness about the wavelength and equal the derivated expression to zero:

$$\frac{\partial B_\lambda}{\partial \lambda} = 0$$

$$\frac{\partial}{\partial \lambda} \left(\frac{2\,hc^2}{\lambda^5} \left(\frac{1}{e^{\frac{hc}{\lambda k_B T}} - 1} \right) \right) = 0$$

$$\frac{-10hc^2}{\lambda^6 \left(e^{\frac{hc}{\lambda k_B T}} - 1 \right)} + \frac{2h^2 c^3 e^{\frac{hc}{\lambda k_B T}}}{\lambda^7 \left(e^{\frac{hc}{\lambda k_B T}} - 1 \right)^2 k_B T} = 0$$

$$5 - 5e^{\frac{-hc}{\lambda k_B T}} = \frac{hc}{\lambda k_B T}$$

Now, we can solve this expression (using the Lambert W equations) for λT and find the Wien's law:

$$\lambda T = const$$

2.2 Einstein's theory of the light-matter interaction

Exercise 22 *Using the relations between the Einstein's coefficients,*

$$g_1 B_{12} = g_2 B_{21} \qquad A_{21} = h \frac{8\pi\nu^3}{c^3} B_{21}$$

, respond:

1. *Can we invert an atomic two-level system?*
2. *Which is the principal difficult to obtain high frequency lasers, for example, x-ray lasers?*

Solution 22

1. A closed two-level system can't be inverted.
2.

Exercise 23 *Which is the validity limit of the Einstein's theory of the light-matter interaction?*

Solution 23

2.3 Rate equations for the populations

Exercise 24 *Determine the conditions in order to an atomic two-level system could amplificate an electric field with the resonance frequency ($|1\rangle \leftrightarrow |2\rangle$), using the rate equations for an open two-level system.*

Solution 24 The rate equations for an open two-level system are:

$$\frac{dN_1}{dt} = -B\rho(\nu)(N_1 - N_2) + A_{21}N_2 + r_1 - a_1 N_1$$
$$\frac{dN_2}{dt} = B\rho(\nu)(N_1 - N_2) - A_{21}N_2 + r_2 - a_2 N_2$$

2.4 Calculation of the B Einstein's coefficient

Exercise 25 *Compare the magnitudes of the electric dipolar interaction and the magnetic dipolar interaction*

Solution 25

Exercise 26

Solution 26

2.5 Two levels system: Exact solution of the RWA

Exercise 27 *Demonstrate that the expressions:*

$$a_1(t) = Ae^{-i(\Delta-\Omega')t/2} + Be^{-i(\Delta+\Omega')t/2}$$
$$a_2(t) = Ce^{+i(\Delta-\Omega')t/2} + De^{+i(\Delta+\Omega')t/2}$$

are the solutions of:

$$\ddot{a_1} + i\Delta\dot{a_1} + \left(\frac{\Omega}{2}\right)^2 a_1 = 0$$
$$\ddot{a_2} - i\Delta\dot{a_2} + \left(\frac{\Omega}{2}\right)^2 a_1 = 0$$

Solution 27

Exercise 28 *Find the figure of the AC-Stark split for a two-level system interacting with electromagnetic wave which $\Delta > 0$*

Solution 28

Exercise 29 *Find explicitly the value of the Rabi frequency (in Hz units) for the experiments showed in the figure 2:*

Solution 29

Exercise 30 *Suppose that you have a two-level atom fallen upon in a cavity with length L that it has an electromagnetic field resonant with the atomic transition. Atom go into the cavity with a velocity v and in the excited state. Which has to be the velocity of the atom to be in the desexcited state when the atom go out the cavity? and in a superposition state $|a_1| = |a_2| = 1/2$?*

Determine the valour of the velocity in the case: $L = 2.7$ cm and $\Omega \cong 40kHz$

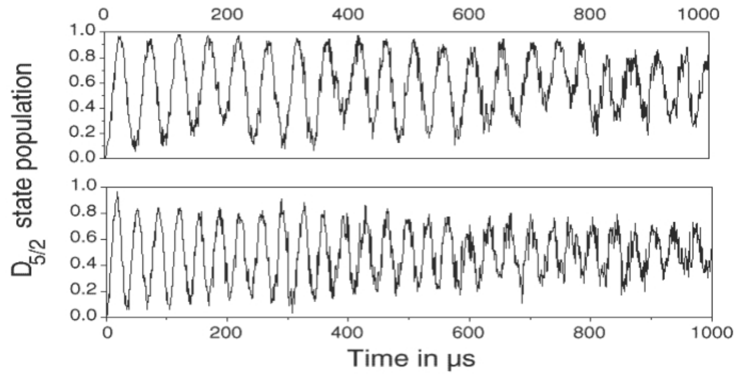

Figure 2: Rabi's oscillations of $^{40}Ca^{+}$

Solution 30

Exercise 31 *Which approximate appearance shows the fluorescent triplet (Mollow triplet) for* $\Omega = 4\Gamma$ *and* $\Delta = \omega_0 - \omega = 3\Gamma$*?*

Solution 31

Exercise 32 *Draw the Autler-Townes doublet spectrum of a prove's field in a three-level system where a powerful field with* $\Omega = 3\Gamma$ *and* $\Delta = 4\Gamma$ *is acting. Consider the scheme corresponding to the V configuration.*

Solution 32

Exercise 33

Solution 33

Exercise 34 *Obtain the result* $N_2^{est} = \frac{n}{2\Gamma}\frac{\Omega^2}{\Omega'^2+\Gamma^2}$*, using the defined integral:* $\int_0^\infty e^{-ax}cos(bx)dx = a/(a^2+b^2)$

Solution 34

2.6 Validity conditions of the Lorentz classical model

Exercise 35 *Obtain the electric dipole* $\mu(t)$ *for a two-level system using the Schrdinger equation for the populations* a_1 *and* a_2

Solution 35

Exercise 36 *Using the movement equations of the amplitude probability of a two-level system, determine the Liouville's operator that we need in the Schrdinger-Von Neumann-Liouville equation.*

Solution 36

Exercise 37 *Prove that the decay of the coherence* ρ_{12} *in an open two-level system (where the relaxation of the two levels falls down outside the two-level system) is* $(\Gamma_1 + \Gamma_2)/2$

Solution 37

3 Quantum theory of the light-matter interaction

3.1 Classical electrodynamics

Exercise 38 *Prove that* $\frac{dH}{dt} = 0$*, where* H *is defined by:* $H = \sum_\alpha \frac{1}{2} m_\alpha \vec{v}_\alpha^2(t) + \frac{\epsilon_0}{2} \int \left\{ \vec{E}^2(\vec{r}, t) + c^2 \vec{B}(\vec{r}, t) \right\} d^3r$

Solution 38

Exercise 39 *Demonstrate the Parseval's identity:* $\int F^*(\vec{r}) G(\vec{r}) d^3r = \int \boldsymbol{F}^*(\vec{k}) \boldsymbol{G}(\vec{k}) d^3k$

Solution 39

Exercise 40 *Demonstrate the property of the Fourier transform of the product of two functions:* $\boldsymbol{F}(\vec{k}) \boldsymbol{G}\vec{k} \leftrightarrow \frac{1}{(2\pi)^{3/2}} \int F(\vec{r}') G(\vec{r} - \vec{r}') d^3r'$

Solution 40

Exercise 41 *Demonstrate the expression:*

$$\vec{B}^* \cdot \vec{B} = \frac{N^2}{c^2} \left(\vec{\alpha}^* \cdot \vec{\alpha} + \vec{\alpha}_-^* \cdot \vec{\alpha}_-^* + \vec{\alpha}^* \cdot \vec{\alpha}_-^* + \vec{\alpha}_- \cdot \vec{\alpha} \right)$$

Solution 41

Exercise 42 *Demonstrate*

Solution 42

Exercise 43

Solution 43

Exercise 44

Solution 44

Exercise 45

Solution 45

3.2 Quantification of the electromagnetic field

Exercise 46 *Find explicitly the commutation relation* $\left[\hat{a}, \hat{a}^\dagger\right]$

Solution 46

3.3 States of the free quantum field

Exercise 47 *Demonstrate explicitly the result:* $\left(\Delta \vec{E}_\perp\right)^2 = \frac{\hbar\omega}{2\epsilon_0 L^3}$

Solution 47

Exercise 48 *Demonstrate the relation of closen between the coherent states:* $\frac{1}{\pi} \int |\alpha\rangle \langle\alpha| \, d^2\alpha = \hat{1}$

Solution 48

3.4 Interaction between atoms and free fields

Exercise 49 *Demonstrate that the elimination of the terms:* $\hat{a}\sigma_-$ *and* $\hat{a}^\dagger\sigma_+$ *in the interaction hamiltonian is equivalent than the application of rotate wave approximation. Help: Consider the operators of the field and the atom in the Heisenberg's picture*

Solution 49

Exercise 50 *Obtain the eigenvalues of the energy, diagonalizating the hamiltonian:* $H_n = \hbar\left(n+\frac{1}{2}\right)\omega \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\hbar}{2} \begin{pmatrix} \Delta & -2ig\sqrt{n+1} \\ 2ig\sqrt{n+1} & -\Delta \end{pmatrix}$

Solution 50