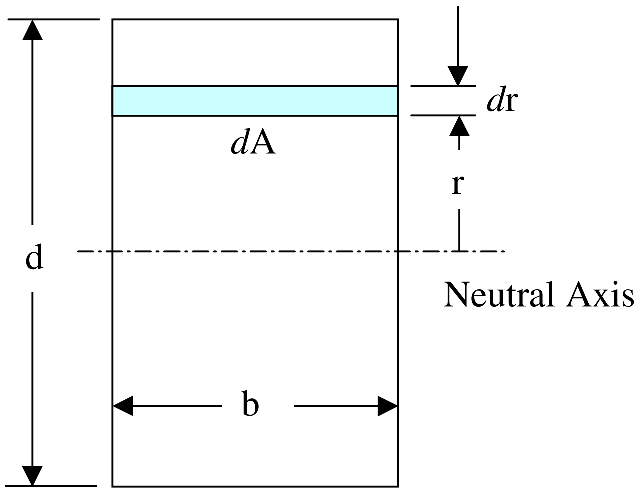
RECTANGULAR SECTION:

Second Moment of Area



$S = I / c$
S = Section Modulus
I = Second Moment of Area
c = Distance from Neutral Axis to Extreme Fiber

$dA = b\,dr$

$$I = \int r^2\,dA = \int_0^{d/2} r^2 b\,dr$$

$$= b\int_0^{d/2} r^2\,dr$$

$$= \frac{b\,(d/2)^3}{3} = \frac{bd^3}{24}$$

The total Second Moment of Area is the sum of the Second Moments of Area on each side of the Neutral Axis.

$$I = \frac{2\,(bd)^3}{24} = \frac{bd^3}{12}$$

Since $S = I / c$, where $c = d / 2$

$$S = \frac{(bd)^3}{12} \div \frac{d}{2} = \frac{bd^2}{6}$$

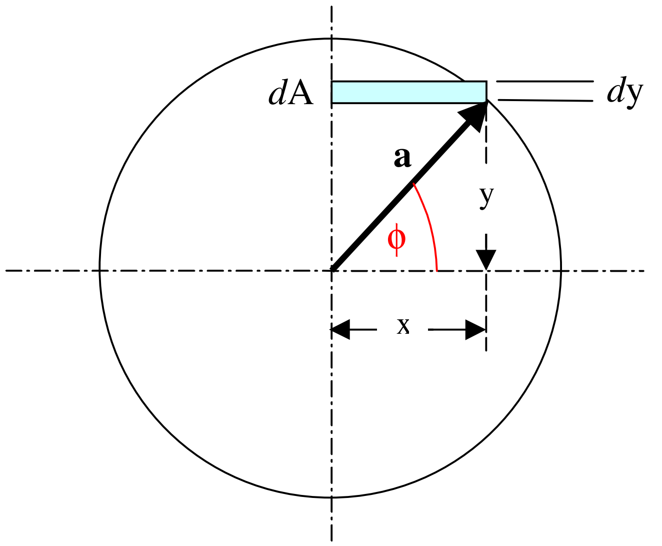
For a square beam, where $b = d$, $S = b^3 / 6$

If the Neutral axis is located at the bottom edge of the section, then:

$$I = bd^3 / 3$$

<u>CIRCULAR SECTION</u>:

Second Moment of Area
Integration along the *y*-axis, using Horizontal Elements



$$I = \int r^2 \, dA$$

$r = y$
$dA = x\, dy$

Due to symmetry, the Total Second Moment of Area is four times the Second Moment of Area of the first quadrant.

From the diagram: $x = a\cos\phi$
$y = a\sin\phi$

$dy / d\phi = d\,(a\sin\phi) / d\phi = a\cos\phi$
$dy = a\cos\phi\, d\phi$

As y varies from zero to a ,
ϕ varies from 0 to $\pi / 2$

$$I = 4\int_0^a y^2\, dA = 4\int_0^a y^2\, x\, dy$$

$$I = 4\int_0^{\pi/2} (a^2 \sin^2\phi)\,(a\cos\phi)\,(a\cos\phi\, d\phi)$$

$$= 4\int_0^{\pi/2} (a^2\sin^2\phi)\,(a^2\cos^2\phi)\, d\phi$$

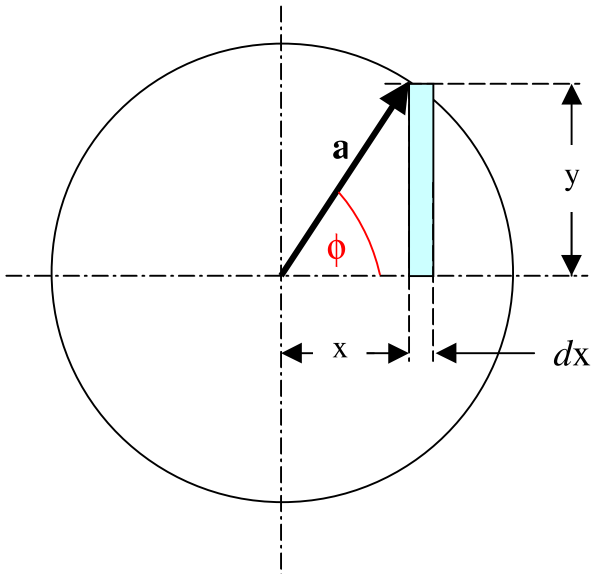
$$= 4a^4\int_0^{\pi/2} \sin^2\phi\, \cos^2\phi\, d\phi = 4a^4\int_0^{\pi/2} \frac{\sin^2 2\phi}{4}\, d\phi \;\ldots\text{(since } \sin\phi\,\cos\phi = \sin 2\phi / 2)$$

$$= a^4\int_0^{\pi/2} \frac{1-\cos 4\phi}{2}\, d\phi \quad \ldots \quad \ldots \quad \ldots \text{(since } \sin^2\phi = (1-\cos 2\phi)/2\,)$$

$$= \left[\frac{a}{2}^4\left(\frac{\pi}{2} - \frac{\sin 4(\pi/2)}{4}\right)\right] - \left[\frac{a}{2}^4\left(0 - \frac{\sin 4(0)}{4}\right)\right] = \frac{a^4\pi}{4}$$

CIRCULAR SECTION:

Second Moment of Area
Integration along the x-axis, using Vertical Elements



Note that for each elemental rectangle, $dI = y^3\, dx / 3$

$$I = \frac{1}{3}\int_0^a y^3\, dx$$

$dx / d\phi = d\,(\cos\phi) / d\phi = -\sin\phi$
$d\,(\cos\phi) = -\sin\phi\, d\phi$

As x varies from zero to a,
φ varies from π / 2 to zero.

$$I = \frac{1}{3}\int_{\pi/2}^{0} (a \sin\phi)^3\, d\,(a\cos\phi)$$

$$= \frac{a^4}{3}\int_{\pi/2}^{0} \sin^3\phi\,(-\sin\phi\, d\phi) = -\frac{a^4}{3}\int_{\pi/2}^{0} \sin^4\phi\, d\phi = \frac{a^4}{3}\int_0^{\pi/2} \sin^4\phi\, d\phi$$

$$I = \frac{a^4}{3}\int_0^{\pi/2} \left(\frac{1 - 2\cos 2\phi + \cos^2 2\phi}{4}\right) d\phi$$

$$= \frac{a^4}{12}\int_0^{\pi/2} \left(1 - 2\cos 2\phi + \frac{1 + \cos 4\phi}{2}\right) d\phi$$

$$= \frac{a^4}{24}\left[3\phi - 2\sin 2\phi + \frac{\sin 4\phi}{4}\right]_0^{\pi/2}$$

$$= \frac{a^4}{96}\left[12\phi - 8\sin 2\phi + \sin 4\phi\right]_0^{\pi/2}$$

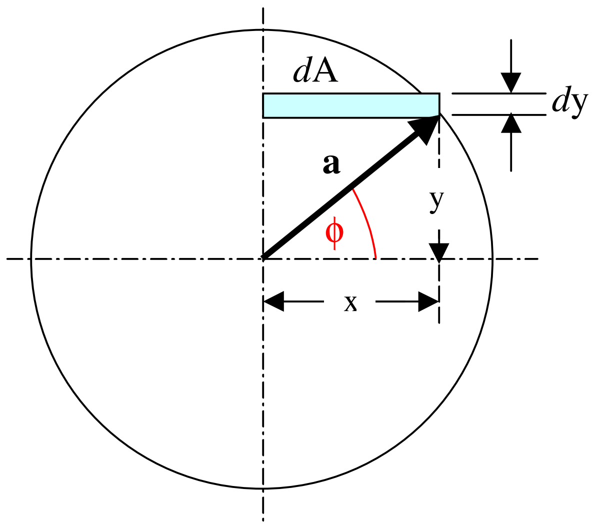
$\sin^4\phi = (\sin^2\phi)^2$,

and $\sin^2\phi = \dfrac{1 - \cos 2\phi}{2}$

$\cos^2\phi = \dfrac{1 + \cos 2\phi}{2}$

CIRCULAR SECTION:

First Moment of Area
Integration along the *y*-axis, using Horizontal Elements



$$A\bar{x} = \int_0^a y\, dA \,,\; dA = x\, dy$$

$$A\bar{x} = \int_0^a yx\, dy$$

As y varies from zero to a
ϕ varies from zero to π / 2

$$A\bar{x} = \int_0^{\pi/2} (a \sin\phi)\,(a \cos\phi)\; d\,(a \sin\phi)$$

$$= a^3 \int_0^{\pi/2} \cos^2\phi \,\sin\phi \; d\phi$$

$dy\, /\, d\phi = d\,(\sin\phi)\, /\, d\phi = \cos\phi$
$d\,(\sin\phi) = \cos\phi\; d\phi$

Let $\cos\phi = u$, $d\,(\cos\phi)\,/\,d\phi = du\,/\,d\phi = -\sin\phi$,
and $-\sin\phi\;\, d\phi = du$

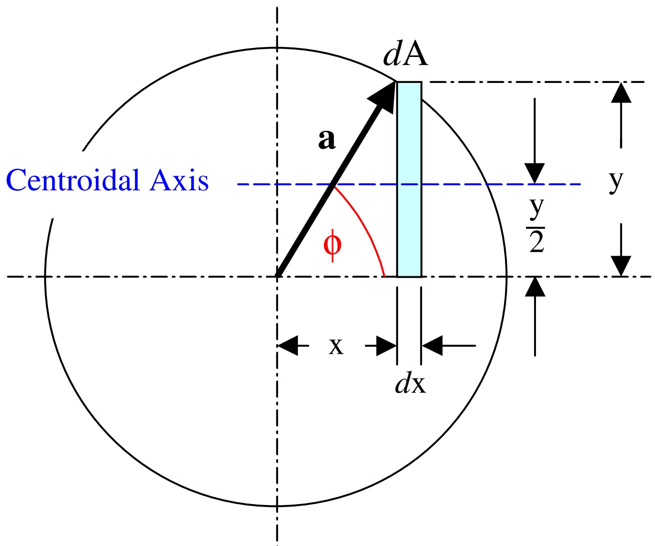
$$A\bar{x} = a^3 \int_{\pi/2}^{0} u^2\, du$$

$$= a^3 \left[u^3 / 3 \right]_{\pi/2}^{0}$$

$$= \frac{a^3}{3} \left[\cos^3\phi \right]_{\pi/2}^{0}$$

CIRCULAR SECTION:

First Moment of Area
Integration along the x-axis, using Vertical Elements



$$A\bar{x} = \int_0^a \frac{y}{2}\, dA$$

$$dA = y\, dx$$

$$A\bar{x} = \int_0^a \frac{y}{2}\, y\, dx$$

$$= \frac{1}{2}\int_0^a y^2\, dx$$

As x varies from zero to a ,
φ varies from π / 2 to 0

$$A\bar{x} = \frac{1}{2}\int_{\pi/2}^{0} (a \sin\phi)^2\, d\,(a\cos\phi)$$

$$= -\frac{a^3}{2}\int_{\pi/2}^{0} \sin^3\phi\, d\phi$$

$dx / d\phi = d(\cos\phi) / d\phi = -\sin\phi$,
$d(\cos\phi) = -\sin\phi\, d\phi$

Let $u = \cos\phi$, $du / d\phi = -\sin\phi$, and $d\phi = du / -\sin\phi$

$$A\bar{x} = -\frac{a^3}{2}\int_{\pi/2}^{0} \sin^3\phi\, \frac{du}{-\sin\phi} = \frac{a^3}{2}\int_{\pi/2}^{0} \sin^2\phi\, du$$

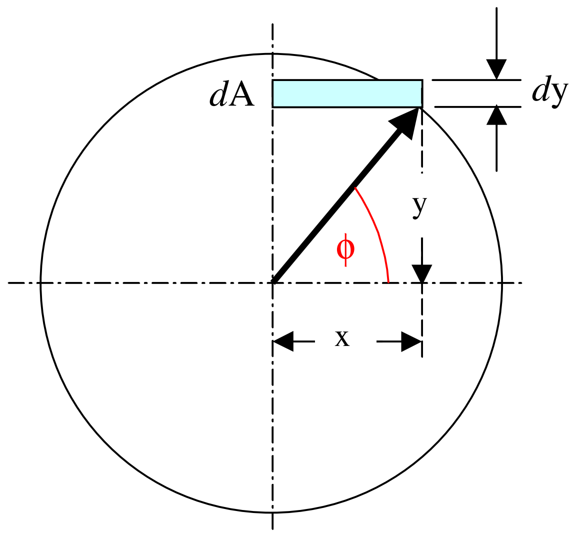
$$= \frac{a^3}{2}\int_{\pi/2}^{0} (1 - \cos^2\phi)\, du = \frac{a^3}{2}\int_{\pi/2}^{0} (1 - u^2)\, du$$

$$= \frac{a^3}{2}\left[u - \frac{u^3}{3}\right]_{\pi/2}^{0} = \frac{a^3}{2}\left[\cos\phi - \frac{\cos^3\phi}{3}\right]_{\pi/2}^{0}$$

CIRCULAR SECTION:

Area

Integration along the y-axis, using Horizontal Elements



$$dA = x\,dy$$

$$A = \int_0^a x\,dy$$

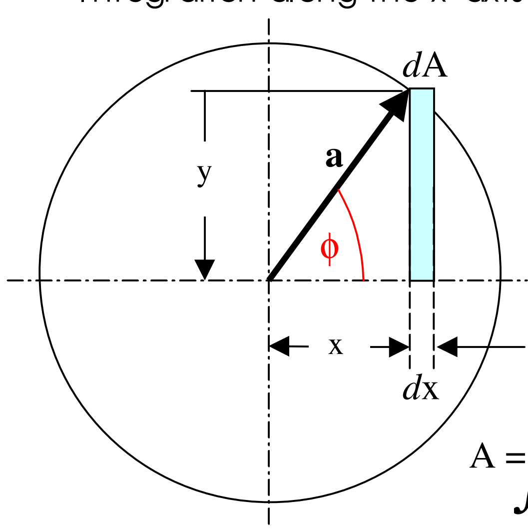
As y varies from zero to a ,
ϕ varies from 0 to π / 2

$dy / d\phi = d\,(\sin\phi) / d\phi = \cos\phi$
$dy = \cos\phi\; d\phi$

$$A = \int_0^{\pi/2} a\cos\phi\;\; d\,(a\sin\phi) \;=\; \int_0^{\pi/2} (a\cos\phi)\,(a\cos\phi)\;d\phi \;=\; a^2\int_0^{\pi/2} \cos^2\phi\;d\phi$$

$$= \frac{a^2}{2}\int_0^{\pi/2} (1+\cos 2\phi)\;d\phi \;=\; \frac{a^2}{2}\left[\phi + \frac{\sin 2\phi}{2}\right]_0^{\pi/2}$$

Integration along the x-axis, using Vertical Elements



$$dA = y\,dx$$

As x varies from zero to a ,
varies from π / 2 to 0

$$A = \int_{\pi/2}^{0} a\sin\phi\;\; d\,(a\cos\phi)$$

$dx / d\phi = d\,(\cos\phi) / d\phi = -\sin\phi$
$d\cos\phi = -\sin\phi\; d\phi$

$$A = \int_{\pi/2}^{0} a\sin\phi\,(-\,a\sin\phi)\,d\phi \;=-\; a^2\int_{\pi/2}^{0} \sin^2\phi\;d\phi$$

$$= \frac{a^2}{2}\int_0^{\pi/2} (1-\;\cos 2\phi)\;d\phi \;=\; \frac{a^2}{2}\left[\phi - \frac{\sin 2\phi}{2}\right]_0^{\pi/2}$$

SUMMARY of ELLIPTIC SECTOR FORMULAS:

The circular sector equations may be modified for elliptic sectors in the event a compound beam comprised of dissimilar materials is required.

Wherever **a** sinϕ is called for, substitute **b** sinϕ; note that **b** is the major axis of the ellipse, and lies on the *y*-axis. The angle defined by ϕ is the parametric angle.

The limits of ϕ are from zero to π / 2; values outside these limits require multiplication by appropriate factors. Integrals are evaluated with respect to the *x*-axis.

Horizontal Elements:

$dA = x\,dy$

$$I = \frac{ab^3}{32}\Big[(4\phi - \sin 4\phi)\Big]_{\phi_1}^{\phi_2}$$

$$A\bar{x} = \frac{ab^2}{3}\Big[\cos^3\phi\Big]_{\phi_2}^{\phi_1}$$

$$A = \frac{ab}{4}\Big[2\phi + \sin 2\phi\Big]_{\phi_1}^{\phi_2}$$

Vertical Elements:

$dA = y\,dx$

$$I = \frac{ab^3}{96}\Big[12\phi - 8\sin 2\phi + \sin 4\phi\Big]_{\phi_1}^{\phi_2}$$

$$A\bar{x} = \frac{ab^2}{6}\Big[3\cos\phi - \cos^3\phi\Big]_{\phi_2}^{\phi_1}$$

$$A = \frac{ab}{4}\Big[2\phi - \sin 2\phi\Big]_{\phi_1}^{\phi_2}$$