Über die anzahl der primzahlen unter einer gegebenen grösse

{On the number of prime numbers less than a given magnitude}

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I believe that I can best convey my gratitude for the honour which the Academy has to a degree conferred on me, through naming me as one of its correspondents, if I immediately avail myself of the privilege thereby received to communicate an investigation into the accumulation of the prime numbers; a subject which perhaps seems not wholly unworthy of such a communication, given the interest which *Gauss* and *Dirichlet* have themselves shown in it over a long period.

In this investigation my point of departure is provided by the observation of *Euler* that the product:

$$\prod \frac{1}{1-\frac{1}{p^s}} = \sum \frac{1}{n^s}$$

where one substitutes for p all prime numbers, and for n all whole numbers. The function of the complex variable s which is represented by these two expressions, wherever they converge, I denote by $\zeta(s)$. They converge only when the real part of s is greater than 1; at the same time it is easy to find an expression for the function which always remains valid. Applying the equation:

$$\int_0^\infty e^{-nx} x^{s-1} dx = \frac{\Pi(s-1)}{n^s}$$

one first sees that:

$$\Pi(s-1)\zeta(s) = \int_0^\infty \frac{x^{s-1}dx}{e^x - 1}$$

If one now considers the integral:

$$\int \frac{(-x)^{s-1}dx}{e^x - 1}$$

from $+\infty$ to $+\infty$ taken in the positive sense around the boundary of a domain which includes the value 0 but no other singularity of the integrand in its interior, then this is easily seen to be equal to:

$$(e^{-\pi s i} - e^{\pi s i}) \int_0^\infty \frac{x^{s-1}dx}{e^x - 1}$$

provided that, in the many-valued function $(-x)^{s-1} = e^{(s-1)\log(-x)}$, the logarithm of $-x$ is determined so as to be real when x is negative. Thus:

$$2\sin \pi s \; \Pi(s-1)\zeta(s) = i \int_\infty^\infty \frac{(-x)^{s-1}dx}{e^x - 1}$$

where the integral has the meaning just stated.

This equation now gives the value of the function $\zeta(s)$ for all complex numbers s and shows that it is single-valued and finite for all finite values of s with the exception of 1, and also that it vanishes if s is a negative even integer.

If the real part of s is negative, then, instead of being taken in the positive sense around the boundary of the given domain, this integral can also be taken in the negative sense around that domain containing all the remaining complex numbers, because then the integral taken through values with infinitely large modulus is infinitely small. But, in the interior of this domain, the integrand has singularities only where x becomes equal to a whole multiple of $\pm 2\pi i$, and the integral is therefore equal to the sum of the integrals taken in the negative sense around these values. Because the integral around the value $n2\pi i$ is $= (-n2\pi i)^{s-1}(-2\pi i)$, one obtains:

$$2\sin \pi s \ \Pi(s-1)\zeta(s) = (2\pi)^s \sum n^{s-1}((-i)^{s-1} + i^{s-1})$$

and therefore a relation between $\zeta(s)$ and $\zeta(1-s)$, which, by using known properties of the function Π, can be formulated as the statement:

$$\Pi\left(\frac{s}{2} - 1\right) \pi^{-\frac{s}{2}} \zeta(s)$$

remains unchanged when s is replaced by $1 - s$.

This property of the function animated me to consider, in place of $\Pi(s-1)$, the integral $\Pi\left(\frac{s}{2} - 1\right)$ in the general term of the series $\sum \frac{1}{n^s}$, whereby one obtains a very convenient expression for the function $\zeta(s)$. In fact:

$$\frac{1}{n^s}\Pi\left(\frac{s}{2} - 1\right) \pi^{-\frac{s}{2}} = \int_0^\infty e^{-n^2\pi x} x^{\frac{s}{2}-1} dx$$

and thus, if one sets:

$$\sum_1^\infty e^{-n^2\pi x} = \psi(x)$$

then it follows that:

$$\Pi\left(\frac{s}{2} - 1\right) \pi^{-\frac{s}{2}} \zeta(s) = \int_0^\infty \psi(x) x^{\frac{s}{2}-1} dx$$

or because:

$$2\psi(x) + 1 = x^{-\frac{1}{2}} \left(2\psi\left(\frac{1}{x}\right) + 1\right) \text{ (Jacobi, Fund. S. 184)}$$

that:

$$\Pi\left(\frac{s}{2}-1\right)\pi^{-\frac{s}{2}}\zeta(s)=\int_1^\infty \psi(x)x^{\frac{s}{2}-1}dx+\int_1^\infty \psi\left(\frac{1}{x}\right)x^{\frac{s-3}{2}}dx+\frac{1}{2}\int_0^1\left(x^{\frac{s-3}{2}}-x^{\frac{s}{2}-1}\right)dx$$

$$=\frac{1}{s(s-1)}+\int_1^\infty \psi(x)\left(x^{\frac{s}{2}-1}+x^{-\frac{1+s}{2}}\right)dx$$

I now set $s=\frac{1}{2}+ti$ and:

$$\Pi\left(\frac{s}{2}\right)(s-1)\pi^{-\frac{s}{2}}\zeta(s)=\xi(t)$$

so that:

$$\xi(t)=\frac{1}{2}-(t^2+\frac{1}{4})\int_1^\infty \psi(x)x^{-\frac{3}{4}}\cos\left(\frac{1}{2}t\log x\right)dx$$

or, in addition:

$$\xi(t)=4\int_1^\infty \frac{d(x^{\frac{3}{2}}\psi'(x))}{dx}x^{-\frac{1}{4}}\cos\left(\frac{1}{2}t\log x\right)dx$$

This function is finite for all finite values of t, and allows itself to be developed as a very rapidly converging power series in t^2. Now, since for a value of s whose real part is greater than 1, $\log\zeta(s)=-\sum\log(1-p^{-s})$ is finite, and since the same holds for the logarithms of the other factors of $\xi(t)$, it follows that the function $\xi(t)$ can only vanish if the imaginary part of t lies between $\frac{1}{2}i$ and $-\frac{1}{2}i$. The number of roots of $\xi(t)=0$, whose real parts lie between 0 and T is approximately:

$$=\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}$$

because the integral $\int d\log\xi(t)$, taken in the positive sense around the region consisting of the values of t whose imaginary parts lie between $\frac{1}{2}i$ and $-\frac{1}{2}i$ and whose real parts lie between 0 and T, is (up to a fraction of the order of magnitude of the quantity $\frac{1}{T}$) equal to $\left(T\log\frac{T}{2\pi}-T\right)i$; this integral is however equal to the number of roots of $\xi(t)=0$ lying within this region, multiplied by $2\pi i$. One now finds indeed approximately this number of real roots within these limits, and it is very likely that all the roots are real. Certainly one would desire a more rigorous demonstration here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the impending objective of my investigation.

If one denotes by α all the roots of the equation $\xi(\alpha) = 0$, then one can express $\log \xi(t)$ as:

$$\sum \log \left(1 - \frac{t^2}{\alpha^2}\right) + \log \xi(0)$$

because, since the density of the roots of the quantity t grows with t only as $\log \frac{t}{2\pi}$, it follows that this expression converges and becomes for an infinite t only infinite like $t \log t$; thus it differs from $\log \xi(t)$ by a function of t^2, which for a finite t remains continuous and finite and which, when divided by t^2, becomes infinitely small for infinite t. This difference is therefore a constant, whose value can be determined by setting $t = 0$.

With these methods, the number of prime numbers that are smaller than x can now be determined.

Let $F(x)$ be equal to this number when x is not exactly equal to a prime number; but let it be greater by $\frac{1}{2}$ when x is a prime number, so that, for any x at which there is a jump in the value in $F(x)$:

$$F(x) = \frac{F(x+0) + F(x-0)}{2}$$

If in the identity:

$$\log \zeta(s) = -\sum \log(1 - p^{-s}) = \sum p^{-s} + \tfrac{1}{2} \sum p^{-2s} + \tfrac{1}{3} \sum p^{-3s} + \cdots$$

one now replaces:

$$p^{-s} \text{ by } s \int_p^\infty x^{-s-1}\, ds, \quad p^{-2s} \text{ by } s \int_{p^2}^\infty x^{-s-1}\, ds, \ldots$$

one obtains:

$$\frac{\log \zeta(s)}{s} = \int_1^\infty f(x) x^{-s-1}\, dx$$

if one denotes:

$$F(x) + \tfrac{1}{2} F(x^{\frac{1}{2}}) + \tfrac{1}{3} F(x^{\frac{1}{3}}) + \cdots$$

by $f(x)$.

This equation is valid for each complex value $a + bi$ of s for which $a > 1$. If, though, the equation:

$$g(s) = \int_0^\infty h(x) x^{-s}\, d \log x$$

holds within this range, then, by making use of *Fourier*'s theorem, one can express the function h in terms of the function g. The equation decomposes, if $h(x)$ is real and:

$$g(a+bi) = g_1(b) + ig_2(b)$$

into the two following:

$$g_1(b) = \int_0^\infty h(x)x^{-a}\cos(b\log x)\,d\log x$$

$$ig_2(b) = -i\int_0^\infty h(x)x^{-a}\sin(b\log x)\,d\log x$$

If one multiplies both equations with:

$$(\cos(b\log y) + i\sin(b\log y))\,db$$

and integrates them from $-\infty$ to $+\infty$, then one obtains $\pi h(y)y^{-\alpha}$ on the right hand side in both, on account of *Fourier*'s theorems; thus, if one adds both equations and multiplies them by iy^α, one obtains:

$$2\pi i h(y) = \int_{a-\infty i}^{a+\infty i} g(s)y^s\,ds$$

where the integration is carried out so that the real part of s remains constant.

For a value of y at which there is a jump in the value of $h(y)$, the integral takes on the mean of the values of the function h on either side of the jump. From the manner in which the function f was defined, we see that it has the same property, and hence in full generality:

$$f(y) = \frac{1}{2\pi i}\int_{a-\infty i}^{a+\infty i}\frac{\log\zeta(s)}{s}y^s\,ds$$

One can substitute for $\log\zeta$ the expression:

$$\frac{s}{2}\log\pi - \log(s-1) - \log\Pi\left(\frac{s}{2}\right) + \sum{}^{\alpha}\log\left(1+\frac{(s-\frac{1}{2})^2}{\alpha\alpha}\right) + \log\xi(0)$$

found earlier; but the integrals of the individual terms of this expression do not converge, when extended to infinity, for which reason it is appropriate to convert the previous equation by means of integration by parts into:

$$f(x) = -\frac{1}{2\pi i}\frac{1}{\log x}\int\limits_{a-\infty i}^{a+\infty i} \frac{d\frac{\log\zeta(s)}{s}}{ds}x^s\,ds$$

Since:

$$-\log\Pi\left(\frac{s}{2}\right) = \lim\left(\sum_{n=1}^{n=m}\log\left(1+\frac{s}{2n}\right) - \frac{s}{2}\log m\right)$$

for $m = \infty$ and therefore:

$$-\frac{d\frac{1}{s}\log\Pi\left(\frac{s}{2}\right)}{ds} = \sum_{1}^{\infty}\frac{d\frac{1}{s}\log\left(1+\frac{s}{2n}\right)}{ds}$$

it then follows that all the terms of the expression for $f(x)$, with the exception of:

$$\frac{1}{2\pi i}\frac{1}{\log x}\int\limits_{a-\infty i}^{a+\infty i}\frac{1}{ss}\log\xi(0)x^s\,ds = \log\xi(0)$$

take the form:

$$\pm\frac{1}{2\pi i}\frac{1}{\log x}\int\limits_{a-\infty i}^{a+\infty i}\frac{d\left(\frac{1}{s}\log\left(1-\frac{s}{\beta}\right)\right)}{ds}x^s\,ds$$

But now:

$$\frac{d\left(\frac{1}{s}\log\left(1-\frac{s}{\beta}\right)\right)}{d\beta} = \frac{1}{(\beta-s)\beta}$$

and, if the real part of s is larger than the real part of β:

$$-\frac{1}{2\pi i}\int\limits_{a-\infty i}^{a+\infty i}\frac{x^s\,ds}{(\beta-s)\beta} = \frac{x^\beta}{\beta} = \int\limits_{\infty}^{x} x^{\beta-1}\,dx$$

or:

$$= \int\limits_{0}^{x} x^{\beta-1}\,dx$$

depending on whether the real part of β is negative or positive. One has as a result:

$$\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-\infty i}^{a+\infty i}\frac{d\left(\frac{1}{s}\log\left(1-\frac{s}{\beta}\right)\right)}{ds}x^s\,ds = -\frac{1}{2\pi i}\int_{a-\infty i}^{a+\infty i}\frac{1}{s}\log\left(1-\frac{s}{\beta}\right)x^s\,ds$$
$$= \int_{\infty}^{x}\frac{x^{\beta-1}}{\log x}\,dx + \text{ const.}$$

in the first, and:

$$= \int_{0}^{x}\frac{x^{\beta-1}}{\log x}\,dx + \text{ const.}$$

in the second case.

In the first case the constant of integration is determined if one lets the real part of β become infinitely negative; in the second case the integral from 0 to x takes on values separated by $2\pi i$, depending on whether the integration is taken through complex values with positive or negative argument, and becomes infinitely small, for the former path, when the coefficient of i in the value of β becomes infinitely positive, but for the latter, when this coefficient becomes infinitely negative. From this it is seen how on the left hand side $\log\left(1-\frac{s}{\beta}\right)$ is to be determined in order that the constants of integration disappear.

Through the insertion of these values in the expression for $f(x)$ one obtains:

$$f(x) = Li(x) - \sum{}^{\alpha}\left(Li\left(x^{\frac{1}{2}+\alpha i}\right) + Li\left(x^{\frac{1}{2}-\alpha i}\right)\right) + \int_{x}^{\infty}\frac{1}{x^2-1}\frac{dx}{x\log x} + \log\xi(0)$$

if in $\sum^{\alpha}$ one substitutes for α all positive roots (or roots having a positive real part) of the equation $\xi(\alpha)=0$, ordered by their magnitude. It may easily be shown, by means of a more thorough discussion of the function ξ, that with this ordering of terms the value of the series:

$$\sum\left(Li\left(x^{\frac{1}{2}+\alpha i}\right) + Li\left(x^{\frac{1}{2}-\alpha i}\right)\right)\log x$$

agrees with the limiting value to which:

$$\frac{1}{2\pi i}\int_{a-bi}^{a+bi}\frac{d\frac{1}{s}\sum\log\left(1+\frac{(s-\frac{1}{2})^2}{\alpha\alpha}\right)}{ds}x^s\,ds$$

converges as the quantity b increases without bound; but when re-ordered it can take on any arbitrary real value.

From $f(x)$ one obtains $F(x)$ by inversion of the relation:

$$f(x) = \sum \frac{1}{n} F\left(x^{\frac{1}{n}}\right)$$

to obtain the equation:

$$F(x) = \sum (-1)^{\mu} \frac{1}{m} f\left(x^{\frac{1}{m}}\right)$$

in which one substitutes for m the series consisting of those natural numbers that are not divisible by any square other than 1, and in which μ denotes the number of prime factors of m.

If one restricts $\sum^{\alpha}$ to a finite number of terms, then the derivative of the expression for $f(x)$ or, up to a part diminishing very rapidly with growing x:

$$\frac{1}{\log x} - 2\sum^{\alpha} \frac{\cos(\alpha \log x) x^{-\frac{1}{2}}}{\log x}$$

gives an approximating expression for the density of the prime number + half the density of the squares of the prime numbers + a third of the density of the cubes of the prime numbers etc. at the magnitude x.

The known approximating expression $F(x) = Li(x)$ is therefore valid up to quantities of the order $x^{\frac{1}{2}}$ and gives somewhat too large a value; because the non-periodic terms in the expression for $F(x)$ are, apart from quantities that do not grow infinite with x:

$$Li(x) - \tfrac{1}{2} Li(x^{\frac{1}{2}}) - \tfrac{1}{3} Li(x^{\frac{1}{3}}) - \tfrac{1}{5} Li(x^{\frac{1}{5}}) + \tfrac{1}{6} Li(x^{\frac{1}{6}}) - \tfrac{1}{7} Li(x^{\frac{1}{7}}) + \cdots$$

Indeed, in the comparison of $Li(x)$ with the number of prime numbers less than x, undertaken by *Gauss* and *Goldschmidt* and carried through up to $x =$ three million, this number has shown itself out to be, in the first hundred thousand, always less than $Li(x)$; in fact the difference grows, with many fluctuations, gradually with x. But also the increase and decrease in the density of the primes from place to place that is dependent on the periodic terms has already excited attention, without however any law governing this behaviour having been observed. In any future count it would be interesting to keep track of the influence of the individual periodic terms in the expression for the density of the prime numbers. A more regular behaviour than that of $F(x)$ would be exhibited by the function $f(x)$, which already in the first hundred is seen very distinctly to agree on average with $Li(x) + \log \xi(0)$.